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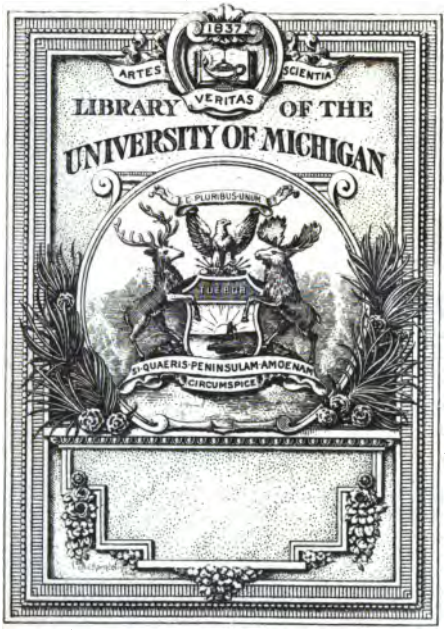
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TRACTS

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ON

MATHEMATICAL

AND

PHILOSOPHICAL SUBJECTS;

COMPRISING,

AMONG NUMEROUS IMPORTANT ARTICLES,

THE THEORY OF BRIDGES,

WITH SEVERAL PLANS OF RECENT IMPROVEMENT.

ALSO

THE RESULTS OF NUMEROUS EXPERIMENTS ON

THE FORCE OF GUNPOWDER,

WITH APPLICATIONS TO

THE MODERN PRACTICE OF ARTILLERY,

IN THREE VOLUMES.

BY CHARLES HUTTON, LL.D. AND F.R.S. &c.

Late Professor of Mathematics in the Royal Military Academy, Woolwich.

VOL. III.

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CONTENTS TO VOL. III.

TRACT XXXIV. continued - - - - - - 1
XXXV. On a new Gunpowder Eprouvette. 153
XXXVI. Determination of the Air's Resistance to Bodies in motion, as determined by the Whirling Machine. - - - - - - 163
XXXVII. On the Theory and Practice of Gunnery, as dependent on the Resistance of the Air. - - - - - - - - 209
XXXVIII. Solution of a Miscellaneous Collection of practical Questions, illustrating several of the preceding Principles. - - - - 316

TRACT XXXIV.

(CONTINUED.)

NEW EXPERIMENTS IN GUNNERY.

THE EXPERIMENTS OF 1784.

68. *Wednesday, July* 21, *&c.* 1784.

IN the course of last year's operations we experienced several inconveniences from some parts of our apparatus, which we determined to remedy if possible. These regarded chiefly the time-pieces, the axes of vibration, and the method of measuring by the tape. For measuring the time of a certain number of vibrations, we united the use of a second stop watch with a simple half-second pendulum, made of a leaden bullet suspended by a silken thread, which did not always agree together. Again, the axes of the gun and pendulum frames were not found to be so devoid of friction as might be wished. But, above all, the chief cause of dissatisfaction, was the method of measuring the extent of the vibrations by means of the tape; which was, notwithstanding all possible care and precaution, still subject to much irregularity, by being wetted by rain, or blown aside by the wind, or otherwise entangled, which rendered the measurements doubtful and irregular.

The preceding part of this year therefore was employed in correcting these and other smaller imperfections in the apparatus. To our time-pieces we added a peculiar one,

which measures time to 40th parts of a second.—Next, by a happy contrivance, the friction of the axes was almost intirely taken off. This was effected by means of sockets of a peculiar construction, for the axes to work in. First imagine the half of a short cylinder, of 2 or 3 inches long, cut lengthways through the axis, and of a diameter a very little more than the ends of the axis that are intended to work in it: if this were all, it is evident that the axis, in vibrating, would touch this socket in one line only, because their diameters were unequal. Next imagine the inside of this socket to be gradually ground down towards each end, from nothing in the middle; so that the inside resembled a tube having its two ends bent downwards, and rising highest in the middle. Then it is evident that the axis will touch the socket in this one middle point only. And further, the under sides of the axis itself were ground a little, to bring the undermost line to a blunt edge, something like the pivots of a scale beam. The consequence was, that the friction was not sensible in a great number of vibrations; and hereafter we commonly made the gun and pendulum vibrate for just 10 minutes, and divided the counted number of vibrations by 10, for the mean number per minute—And for measuring the arcs of vibration more certainly and accurately, we constructed a strong wooden circular arch, of about 4 feet in length, cut out to a radius of just 10 feet. This arch is divided into chords of equal parts, each the 1000th part of the radius, or $\frac{12}{100}$th parts of an inch, as before described in Art. 16. This arch being placed 10 feet below, and concentric with the axis, and the groove in the middle of it filled with the soft composition of soap and wax, the stylette, or small sharp spear, traces in the groove the extent of the vibration, and the corresponding divisions on each side of the groove show the length of the chord vibrated. And as these chords are in 1000th parts of the radius, the value of r, in the theorem for the velocity of the ball, will be 1000 for all the following experiments; and then that theorem will become

$$v = \frac{59}{96} \times \frac{p+b}{bin} gc \text{ by the pendulum, and}$$

$$v = \frac{59}{96} \times \frac{G g c}{bin} \text{ by the recoil of the gun.}$$

Or $v = 12{\cdot}742 \times \frac{c}{b}$ or $\frac{51}{4} \times \frac{c}{b}$ by the gun n° 2, when we substitute the values of G, g, i, n, specified in Art. 36. And further, when $b = 1{\cdot}047$, it is $v = 12\frac{1}{6}c$.

The apparatus having been prepared, we employed the three days, July 21, July 26, and August 3, in hanging it up, and in weighing, measuring, and adjusting all the parts, and trying them by firing a few rounds with powder only. The 4 rounds fired on the first of those days, of 4 ounces each, with the gun n° 1, weighing 917 lb, gave 56 at the first round, and at each of the other three 57 divisions on the measuring arc, for the recoil of the gun.

69. *Wednesday, August* 4, 1784.

Frequent showers of rain.

N°	Powder	Weight of gun	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz	lb			inches	in	lb	inches		feet
1	2	478	55							
2	2	478	55							
3	2	478	57							
4	2	478	57							
5	4	478	122							
6	8	478	252							
7	6	478	426	166	88·9	9	631·5	76·79	40·23	1313
8	6	478	387	151	89·3	5	632·8	76·81	40·23	1192
9	6	478	426	164	88·9	8	634·0	76·83	40·23	1303
10	6	650	279	158	89·2	7	635·3	76·85	40·23	1254
11	6	650	290	160	88·0	8	636·6	76·87	40·23	1291
12	6	917	193	157	87·3	9	637·9	76·90	40·22	1280
13	6	917	199	162	86·9	9	639·1	76·92	40·22	1329
14	6	917	85	0						
15	6	917	104	20	82·3		640·1	76·94	40·22	
									Omitting n° 8, the mean is	1295

The general weight of the ball was 16 oz 14 dr; except n° 15, which was 1 oz 10 dr; and n° 14 was without any ball.

Here, and in all the future days, the chords of vibration, of both gun and pendulum, are expressed in 1000th parts of the radius.

The GUN was n° 1.—We began this day with the weight of the gun and its iron frame only, without any of the leaden weights. Then the one set of weights was put on at n° 10, and the other at n° 12. This was done to try the effects of different weights of gun on the velocity of the ball, experimentally to correct a common error which had been adopted from time immemorial, by professional men, namely, that heavier guns, *cæteris paribus*, give the greater velocities. The erroneousness of which opinion is proved by the experiments of this and some of the following days. And it is needless to prove *a priori* to scientific men, that the difference in the effects cannot be rendered sensible by any measurements which we can make of the velocity.

The PENDULUM was the block of last year, with a new core, and a facing of sheet lead. Its weight, taken this morning, was 627 lb.

The plugs weighed 7 ounces to 11 inches, on an average; which proportion may always be used in future, at least till another be mentioned.

The 8th n° is doubtful, and is omitted in the medium.

The 14th was with powder only, like the first six. And the 15th was without ball, having only a wad made of junk, weighing 1 oz 10 dr. This made a small impression, of about half an inch deep, in the face of the pendulum, and rebounded back. And it struck the pendulum at more than 6 inches above the line of direction.

Note, the centre of the pendulum, as before, is at 88·7 inches below the axis. And the value of i, for the mean distance of the points struck, is 88·4.

By comparing together the first 6 rounds, which are all with the same weight of gun, we find that the mean

proportion of the recoil, with the different charges, without balls, is as follows:

2 oz	4 oz	8 oz
56	122	252

the recoils being rather in a higher proportion than the charge of powder.

If we compare the mean of the first 4, with 2 oz of powder and 478 lb weight of gun, with the mean of July 21, with 4 oz of powder and 917 lb weight of gun, we shall obtain as follows:

Charge - - -	2 oz	4 oz
Weight of gun	478 lb - -	917 lb
Recoil - - -	56 - -	57

So that, in this instance, the less charge gives a recoil, in proportion to the greater charge, a little above the direct ratio of the weight of powder, and inverse ratio of the weight of the gun. For that ratio, or 2 × 917 to 4 × 478, is as 56 to 58.

If we compare n° 5 with the mean of July 21, which are both with 4 oz of powder, they will stand thus:

Weight of gun	478 - -	915
Recoil - -	122 - -	57

which shows that, in this instance, the same charge gives more than double the recoil to half the weight of the gun.

Lastly, if we compare the means of each pair of velocities with the several weights of gun, we shall have as follows:

1313, 1303	1308 mean with	478 lb wt of gun
1254, 1291	1273 - -	650 - - -
1280, 1329	1305 - -	917 - - -

which differences are neither regular, nor greater than what happen from different trials, with the weight and all other circumstances the same.

For the 6 oz charge the

Mean recoil with ball -	196
Ditto without - - -	85
Difference, or c = - -	111
Hence velocity by recoil	1339
Ditto by the pendulum	1295
Difference - - - -	44
Or the part - - - -	$\frac{1}{30}$

70. *Thursday, August* 5, 1784.

A fine warm day.

Barometer 29·98; Thermometer 68 at 10 A. M.

N°	Weight of gun	Vibration of gun	Vibration of pend	Point struck	Plgus	Values of p	Values of g	Values of n	Veloc. of ball
	lb			inches	inc	lb	inches		feet
1	485	127							
2	485	176							
3	485	460	195	86·8	9	640·4	76·94	40·22	1606
4	485	459	197	87·2	8	641·7	76·96	40·22	1619
5	655	312	196	88·1	7	642·9	76·99	40·22	1598
6	655	319	196	88·3	7	644·2	77·01	40·22	1598
7	917	218	200	87·3	9	645·5	77·03	40·21	1653
8	917	216	196	88·3	9	646·7	77·06	40·21	1605
9	1170	174	202	89·4	7	648·0	77·08	140·2	1637
10	1170	168	198	89·1	8	649·3	77·11	140·2	1614
								mean of all	1616

The weight of every ball was 16 oz 14 dr. The first charge of powder was 4 ounces, and all the rest 16 ounces.

The GUN was n° 3.—Began first with its own weight only; then at n° 5 put on one pair of the usual weights; at n° 7 the other pair; and lastly at n° 9 fixed on some extra weights. But the result shows that the velocity of the ball is the same with all of them.

The PENDULUM as left hanging since yesterday.

The value of i, or medium among the points struck these last two days, is 88·2.

1606 } 1613 mean velocity with 485 lb weight of gun
1619 }
1598 } 1598 - - - - - - 655 - - - - - -
1598 }
1653 } 1629 - - - - - - 917 - - - - - -
1605 }
1637 } 1625 - - - - - - 1170 - - - - - -
1614 }
1616 mean for 6 oz with gun 3.

71. *Saturday, August* 7, 1784; *from* 11 *till* 2.

The weather fair, but cloudy at times.

Barometer 29·92; Thermometer 64° at 2 P. M.

N°	Weight of wad		Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n
	oz	dr			inches	in	lb	inches	
1			58						
2	2	10	119	31	93·0				
3	2	9	120	20	78·0				
4			93						
5			93						
6	2	10	225	206	89·7	4	651·6	77·14	40·21
7	2	8	228	212	89·4		652·9	77·16	40·21
8	2	9	230	206	87·6	5	654·1	77·19	40·21
9	2	9	232	217	89·8	12	655·4	77·21	40·21
10	2	8	231	206	88·8	10	656·7	77·23	40·20
11			219	199	89·2	6	657·9	77·26	40·20
12	4	14	236	205	89·7	8	659·2	77·28	40·20
13	4	12	235	220	89·5	5	660·5	77·30	40·20

The first charge of powder was 4 ounces, and all the rest 6 oz; the balls 16 oz 14 dr.

The GUN, n° 3, with the usual leads, weighed 917 lb.

The mean height of the charge of 6 ounces was 4·5 inches.

The PENDULUM as left hanging since the last day.

The value of i, or the mean among the points struck these last three days, 88·5.

The object of this day's business, was to try the effect of

different degrees of ramming the charge of powder, with the effect of wads placed in different positions. Sometimes the powder was only set up without being compressed, and sometimes it was rammed with a different number of strokes, and pushed with various degrees of force: but no sensible difference was produced in the velocity. The wads, which were of 2 inches length, firmly made of junk or rope yarn, and made large to be with difficulty pushed into the gun, were diversly placed and varied in number, being sometimes introduced between the powder and ball, and sometimes over both. But no effect was perceived from them on the velocity of the ball; this being indifferently the same, either with one wad, or two, or none at all. The reason of which is probably because the balls had very little windage. At the last two numbers two wads were used; in most of the others only one; weighing on an average about 2 oz. 9 dr.

When balls were used with the wads, it was common for them both to enter the pendulum by the same hole. But it is remarkable that, when the wads were discharged without balls, they commonly struck wide of the line of the gun by 6 or 8 inches, and indifferently either too high or too low, or to the right or left; and sometimes they flew in pieces before they struck the block.

The velocities of the ball in these experiments are not computed, as the effects of the blow from the ball and the wad are compounded together, and that in an unknown degree, as the wad sometimes flies in pieces, and sometimes not, or strikes the pendulum with various degrees of force at different times; and also sometimes the wads enter the pendulum, and sometimes they rebound from it.

72. *Tuesday, August* 10, 1784; *from* 12 *till* 2.

The weather thick and cloudy.

N°	Weight of ball		Weight of wad		Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n
	oz	dr	oz	dr			inches	in	lbs	inches	
1					56						
2					56						
3	14	2½			203	170	89·4	7	674·8	76·62	40·26
4	14	3			195	159	89·8	8	676·1	76·64	40·26
5	14	2½	4	12	214	161	89·5	5	677·3	76·67	40·26
6	14	3½	4	10	215	163	89·3	6	678·6	76·69	40·26
7	14	4	2	5	208	171	89·3	6	679·9	76·72	40·26
8	14	2½	2	4	208	171	89·1	6	681·1	76·75	40·25
9	13	15½	2	10	208	176	89·5	5	682·4	76·77	40·25
10	14	3½	4	10	230	187	90·3	8	683·7	76·80	40·25
11	14	3½	4	12	232	218	90·8	7	685·0	76·82	40·25

N°s 1 and 2 were with 4 oz of powder; all the rest 6 oz.

The mean diameter of the ball was 1·875; so that the windage was ·15.

The mean height of the charge of powder was 4·4.

The GUN n° 3; its weight 917 lb.

The object this day was the effect of windage with low balls, and the effect of wads, both high and low ones. The wads struck variously, either above or below or with the ball. The two wads in the last round were made of well-twisted twine, and firmly bound: they struck the pendulum very hard blows. The other wads were of junk, and did not strike so hard.

Here, the balls being smaller, and consequently the windage more, the vibrations are much smaller, though wads were used. So that it seems the wads do not prevent the escape of the inflamed powder by the windage, nor make any sensible alteration in the velocity of the ball.

The velocities are not computed, for the same reason as specified in the last day's experiments.

73. The PENDULUM block had not been altered since the last day's experiments. But the iron stays of the stem had

been changed for others that are stronger, and which weigh 10 lb more than the old ones did. This additional 10 lb of iron must be added to the weight of the pendulum; and new theorems must be made out, for determining the change in the centre of gravity, and the number of vibrations per minute. Now this rod, of uniform thickness, reached from the lower side of the axis to within 24 inches of the top of the block; consequently its length was 51·4 inches, and its middle point, or centre of gravity, was at 26·6 inches below the middle of the axis of vibration. This number 26·6 then will be the value of i in the theorem $G = 77{\cdot}3 + \frac{i - 77{\cdot}3}{660 + b} b$, for the place of the new centre of gravity, where the value of b is 10; which theorem gives

$$G = 77{\cdot}3 - 0{\cdot}76 = 76{\cdot}54 \text{ for the centre of gravity.}$$

And the same values of i and b, substituted in the theorem

$$N = 40{\cdot}2 - \frac{40{\cdot}2\, bi(i - 86{\cdot}3)}{8805707 + bi(i + 86{\cdot}3)}, \text{ give}$$

$$N = 40{\cdot}2 + {\cdot}07 = 40{\cdot}27 \text{ for the number of oscillations.}$$

Hence then, in this new state of the pendulum, the value of g is 76·54, and the value of n 40·27, corresponding to the value 670 of p, or weight of the pendulum. That is,

p	g	n
670	76·54	40·27

are the new radical corresponding values of p, g, n. And these values, being substituted in the two general theorems, namely,

$$G = g + \frac{i - g}{p + b} b, \text{ and}$$

$$N = n - \frac{bin(inn - 140850)}{281700\, pg + bi(inn + 140850)},$$

they become

$$G = 76{\cdot}54 + \frac{i - 76{\cdot}54}{670 + b} b, \text{ and}$$

$$N = 40{\cdot}27 - \frac{40{\cdot}27\, bi(i - 86)}{8820313 + bi(i + 86)},$$

or $N = 40{\cdot}27 - \frac{i - 86}{217947} bi$ nearly. Which are the theorems to be used now and hereafter for the values of g and n. And where the distance of the centre of oscillation, answering to the number 40·47, is 86.

The value of *i* this day, or the mean distance of the points struck, is 89·7.

74. *Wednesday, August* 11, 1784; *from* 10 *till* 2.

The air was warm, close, and thick.

Barometer 30·25; Thermometer 65° at 10 A. M.

N°	Powder	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of *p*	Values of *g*	Values of *n*	Veloc. of ball
	oz			inches	in.	lb	inches		feet
1	4	D 65							
2	10	272	183	89·4	7	686·7	76·87	40·25	1561
3	8	240	176	89·7	6	688·0	76·86	40·24	1499
4	12	297	183	89·5	6	689·3	76·91	40·24	1566
5	10	258	168	88·8	7	690·6	76·93	40·24	1452
6	10	263	168	88·1	6	691·8	76·95	40·24	1466
7	12	293	177	88·2	8	693·1	76·97	40·24	1546
8	14	327	182	89·8	10	694·3	76·99	40·24	1565
9	14	310	175	89·8	9	695·5	77·01	40·24	1508
10	8	240	172	88·6	8	696·9	77·03	40·23	1505
11	8	232	164	89·7	8	698·1	77·05	40·23	1421
12	6		151	89·5	6	699·4	77·07	40·23	1319
13	6	200	157	88·6	6	700·7	77·09	40·23	1388
14	12	283	157	85·1	9	702·0	77·11	40·23	1448
15	14	318	169	88·0	7	703·2	77·13	40·23	1510
16	6	202	149	84·0	8	704·5	77·15	40·23	1398

The diameter of the balls 1·965, and weight 16 oz 14½ dr.

The GUN was n° 1, weighing, with the usual leads, 917 lb.

The PENDULUM as left hanging since yesterday.

The mean value of *i*, for the last two days, is 88·91.

After the experiments were ended this day, the pendulum was weighed, and found to be 706 lb. Now the original weight, when weighed at first on the 26th of July, seemingly with as much care as now, was 627 lb; to this add 61½ lb weight of balls and plugs lodged in it, with 10 lb of iron added on the 8th of August, and they make together 698½ lb; from this take 1·6 lb, for the diminution of the leaden facing of the pendulum, by the balls striking and piercing it, and

there will remain only 697lb, which the pendulum ought to weigh, and which is 9lb less than it is actually found to weigh. I cannot imagine any cause to which this difference of weight may be attributed, as it is contrary to the effect heretofore experienced, the pendulum having always been found to lose in weight by hanging up; unless it arise from the moisture imbibed by the block in the 17 days it was up, and during all or the most part of which time it was very rainy weather, and the pendulum hung uncovered. Now the probability of this will be heightened by considering that the block had lain by all the preceding winter, and till after midsummer this year, under cover, in the carpenter's shop, a circumstance which would make it very dry, and so render it apt to imbibe moisture from the continually foggy atmosphere and rain, which have taken place ever since it was exposed. This increase of weight then, being 9lb in the 17 days, or nearly half a pound per day, I have thought it safest to divide equally among all the days, by adding half a pound for each day it hung up, from the beginning of this year to the end of this day's experiments.

The object of this course was again to search out the maximum of the gun's charge; but it is not a good set of experiments, the velocities being not regular, perhaps owing to the bad state of the pendulum, which was very much shattered. However it sufficiently appears that there was but little difference among the velocities due to 8, 10, 12, and 14 ounces of powder.

Weight of Powder	6 oz	8 oz	10 oz	12 oz	14 oz
Mean height of charge	4·4	5·7	7·0	8·1	9·5
Mean recoil of gun -	201	237	264	291	318
Velocities by pendulum - - - -	1319	1499	1561	1566	1565
	1388	1505	1452	1546	1508
	1398	1421	1466	1448	1510
Mean ditto - - -	1368	1475	1493	1520	1528

75. *Thursday, September* 9, 1784.

Since the last experiments, the steadying-rods of the gun-frame having been lengthened, and the pendulum block repaired with a new core, &c, we attended to weigh and measure the several parts; the circumstances of which were as follow:

Weight of the pendulum -	638	$= p$
Therf. to its centre of gravity	75·93	$= g$
And its vibrations per minute	40·30	$= n$

The new stay-rods of the gun-frame weigh 17lb more than the old ones, so that now

	lb	
The weight of iron in the frame is	205	
Weight of gun and iron together	495	
Weight of gun, iron, and leads -	934	$= o$

By this additional 17 lb weight of iron, the values of g and n, or the centre of gravity and number of oscillations, will be altered; which will cause an alteration in our theorem $v = \frac{59}{96} \times \frac{ogc}{bin}$, by which the velocity of the ball is determined from the recoil of the gun, in Art. 36. The values of these two letters were, at Art. 42 and 43, found to be $g = 80{\cdot}47$, and $n = 40{\cdot}0$ for the gun n° 2; but the former will now become something less, and the latter something greater.

Now the old and new iron stay-rods were nearly of equal thickness. But the old rods extended only 29 inches, and the new ones 58 inches below the axis; the difference is 29; and the half difference, or $14\frac{1}{2}$ added to the old length 29, gives $43\frac{1}{2}$ inches below the axis, where the middle or centre of gravity of the additional length is situated, the weight of which part is 17 lb. But the centre of gravity was found to be 80·47 below the axis, when the whole weight was 917 lb. Here, the difference of the two distances, or the distance between the two weights 17 and 917, being 37 inches, and

the sum of the weights 934, we shall have 934 : 17 : : 37 : 0·67 the change in the distance of the centre of gravity; which being subtracted from 80·47, leaves 79·8 for the distance of the new compound centre of gravity.

Also the correction for the value of n will be determined by the usual formula $\frac{bin(inn - 140850)}{281700\,pg + bi(inn + 140850)}$, in which $b = 17$, $i = 43·5$, $n = 40·0$, $p = 917$, and $g = 83·47$; which values, being used in that formula, give 0·1 for the correction of n; to which add 40·0, and we shall have 40·1 for the new value of n, or number of oscillations per minute, for the gun n° 2; and consequently 40·2 for n° 1, and 40·0 for n° 3, and 39·9 for n° 1. Hence then the new values for the gun n° 2 are thus:

G	g	n	i	r
934	79·8	40·1	89·15	1000

Then, using these values of G, g, n, i, r, in the formula $v = \frac{59000}{96} \times \frac{G\,g\,c}{b\,i\,r\,n}$ above-mentioned, it becomes $v = \frac{205}{16} \times \frac{c}{b}$ for the velocity by the recoil of the gun; where b is the weight of the ball, and c the difference between the chords of recoil with and without a ball.

And when $b = 1·047$ lb $= 16$ oz 12 dr, the same theorem is $v = 12\frac{1}{3}c$ for the gun n° 2. And every $\frac{1}{2}$ dram in the value of b will alter this theorem by the $\frac{1}{525}$th part nearly.

Also for the gun n° 1 the above velocity must be decreased by the 400th part, and for n° 3 increased by the 400th part, and for n° 4 increased by the 200th part.

76. *Friday, September* 10, 1784; *from* 10 *till* 1.

The weather fair; but not warm.

N°	Powder	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	oz			inches	in.	lb	inches		feet
1	4	115							
2	4	116							
3	6	194							
4	6	190							
5	6	193							
6	4		143	88·0	8	638·0	75·93	40·30	1148
7	4	322	140	88·3	7	639·2	75·95	40·30	1123
8	4	324	144	89·3	7	640·3	75·97	40·30	1144
9	4	318	138	88·4	6	641·5	75·99	40·30	1110
10	6	433	173	88·5	7	642·7	76·02	40·30	1393
11	6	432	173	89·9	6	643·8	76·04	40·30	1374
12	6	430	172	90·1	7	645·0	76·06	40·30	1366
13	6	427	168	89·6	6	646·1	76·09	40·30	1345
14	8	519	188	90·0	7	647·3	76·11	40·30	1501
15	8	498	172	88·9	8	648·5	76·13	40·30	1394 D
16	8	529	190	89·6	6	649·6	76·15	40·30	1530
17	2		92	89·4	3	650·8	76·17	40·29	744
18	2	197	98	90·3	3	652·0	76·19	40·29	786
19	2	187	91	89·8	3	653·1	76·21	40·29	736

The ball's diameter 1·96, and weight 16 oz 12 dr.

The GUN, n 1, without the leaden weights, weighed 495 lb.

The PENDULUM as specified the last day.

The plugs weigh 6 ounces to 7 inches long, not being of so dry wood as before. And this rate of the weight of the plugs to be continued till an alteration is announced.

The mean value of i, or point struck, is 89·29.

Here 439 lb weight of lead being taken off, at the distance 90·3 below the axis; and the centre of gravity yesterday being at 79·8 distance, when the whole weight was 934 lb; therefore 495 : 439 :: 10·5 : 9·3 the change in the centre of gravity; and consequently $79{\cdot}8 - 9{\cdot}3 = 70{\cdot}5 = g$ is the distance of the new centre of gravity for this day.

Also the new number of oscillations per minute for this day will be found by this formula,

$$375{\cdot}3\sqrt{\frac{\text{G}g-bi}{\text{G}go-bii}} = 375{\cdot}3\sqrt{\frac{\text{G}g-bi}{\text{G}g\,.\,\frac{140850}{nn}-bii}} = 40{\cdot}5;$$

where the values of the letters are thus, namely:

$$\begin{aligned} \text{G} &= 934 \\ g &= 79{\cdot}8 \\ b &= 439 \\ i &= 90{\cdot}3 \\ n &= 40{\cdot}2 \end{aligned}$$

Now in this day's experiments, the

Charge or weight of Powder	2 oz	4 oz	6 oz	8 oz
Mean height of ditto -	1·8	3·1	4·3	5·8
Mean recoil with ball -	192	321	431	515
Ditto without - - - - -	.	115	192	.
Difference, or c = - -	.	206	239	.
Hence velocity by recoil	.	1170	1358	.
Mean ditto by pendulum	755	1131	1370	1475
Difference - - - - - -		+39	−12	
Or nearly the part - -		$\frac{1}{30}$	$\frac{1}{114}$	

These velocities from the recoil are found by the theorem $\frac{59}{96} \times \frac{\text{G}gc}{bin}$, where the values of the letters are thus:

$$\begin{aligned} \text{G} &= 495 \\ g &= 70{\cdot}5 \\ b &= 1{\cdot}047 \\ i &= 89{\cdot}15 \\ n &= 40{\cdot}5 \end{aligned}$$

77. *Saturday, September* 11, 1784; *from* 10 *till* 1.

Very hot and clear weather.

N°	Ball's diam	Ball's wt	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. ball
	inches	oz dr			inches	n	lb	inches		feet
1			58							
2	1·97	16 14	249	225	89·6	10	654·3	76·23	40·29	1814
3	1·92	16 4	248	206	89·6	6	655·4	76·25	40·29	1730
4	1·87	15 2	241	185	88·7	8	656·6	76·27	40·29	1693
5	1·97	16 14	262	224	89·7	3	657·8	76·29	40·29	1815
6	1·92	16 2	249	201	88·9	7	658·9	76·32	40·28	1725
7	1·87	15 2	236	177	88·6	7	660·1	76·34	40·28	1631
8	1·97	16 14	165	165	90·0	7	661·3	76·36	40·28	1341
9	1·92	16 2	155	146	89·3	6	662·4	76·39	40·28	1255
10	1·87	15 2	149	134	89·7	7	663·6	76·41	40·28	1228
11	1·97	16 14	165	165	89·9	6	664·8	76·43	40·28	1351
12	1·92	16 1	153	142	89·[illegible]	5	665·9	76·45	40·27	1283
13	1·87	15 2	146	132	89·3	5	667·1	76·48	40·27	1222

The charge of powder n° 1 was 4 oz; nos 2, 3, 4, 5, 6, 7 each 8 oz; the rest 4 oz.

The GUN was n° 3, and weighed 934 lb.

At n° 2 the recoil 249 of the gun is too small; owing to the stylette, which ought to trace the arc, not marking all the way.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck, these two days, is 89·34.

The object this day was the effect of different sizes and weights of balls, and different degrees of windage.

The mean weight of balls and velocity, for the two weights of powder 4 and 8 ounces, are as follow:

Powder's wt	Powder's ht	Ball's wt oz	Ball's wt dr	Ball's diam inches	Recoil gun	Veloc. ball
4	3·4	15	2	1·87	148	1225
.	.	16	2	1·92	154	1244
.	.	16	14	1·97	165	1346
8	5·9	15	2	1·87	239	1662
.	.	16	3	1·92	249	1728
.	.	16	14	1·97	262	1815

Here the decrease of the velocity is uniformly observable with the decrease of weight in the ball, and that in a very considerable degree, instead of increasing, which it ought to do, if the windage were the same, or the balls had the same diameter, and that in the reciprocal subduplicate ratio of the weight of the ball. Now that ratio is the ratio of $\sqrt{15\frac{1}{3}}$ to $\sqrt{16\frac{7}{8}}$, or of 11 to $11\frac{7}{17}$. Therefore, as $11 : 11\frac{7}{17} :: 1346 : 1424$ the velocity the least ball would have had, if its diameter had been equal to the heaviest. But its velocity was actually no more than 1225; and therefore the difference 199, or $\frac{1}{7}$ of the whole, or $\frac{1}{6}$ of the experimented velocity, is the velocity lost by the difference of windage; though this difference was only $\frac{1}{10}$ of an inch, or $\frac{1}{20}$ of the caliber, which is no more than the usual windage allowed in service. But the force, or inflamed powder, lost by the same cause, will be $\frac{3}{7}$, or a double part of the velocity, because the velocity is as the square-root of the force or quantity of powder. Hence then, in charges with 4 ounces of powder, and a windage of $\frac{1}{20}$ of the caliber, $\frac{3}{7}$ of the charge is lost, or nearly a mean between $\frac{1}{3}$ and $\frac{1}{4}$.

And if the computation be made in like manner for the above charges of 8 ounces of powder, it will be found that the part of the charge lost by the same windage, will be, in the case of 8 ounces, $\frac{4}{15}$ of the whole; which is still more than the $\frac{1}{4}$ part, though somewhat less than in the case of 4 ounces. The reason of which is, that the ball is sooner out

of the gun with the 8 oz charge, and so the fluid has less time to escape in.

78. *Thursday, September* 16, 1784.

To try the effect of firing the charge of powder in different parts of it.

N°	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of s	Values of n	Veloc. of the ball
1	347	160	88·0	4	668·3	76·50	40·27	1388
2	348	161	87·8	7	669·4	76·52	40·27	1387
3	353	165	88·5	6	670·6	76·54	40·27	1413
4	350	161	87·5	6	671·8	76·57	40·27	1398
5	346	157	87·3	5	672·9	76·59	40·27	1369
6	352	159	87·2	5	674·0	76·61	40·27	1390

The powder 4 oz; ball's diameter, 1·96 inches; and weight 16 oz 9 dr.

The GUN was n° 3; its weight 500 lb.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck, these 3 days, 89·03.

Powder wt	Powder ht	Recoil gun	Mean veloc. of the ball
4	3·1	34·9	1391

The cartridge of n°s 1, 2, and 4 was fired at the fore part; n°s 3 and 5 behind; and n° 6 in the middle: but there does not appear to be any difference among them.

79. *Tuesday, September* 21, 1784; *from* 10½ *till* 1½.

The weather moderately warm.

N°	Powder.	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. feet
	oz			inches	inc	lb	inches		ball
1	4	166							
2	4	457	132	89·7	5	683·0	76·66	40·28	1125
3	4	451	132	91·3	4	684·1	76·68	40·28	1107
4	4	458	136	91·5	4	685·2	76·70	40·28	1140
5	6		157	88·9	4	686·3	76·72	40·28	1357
6	6	613	162	91·0	4	687·5	76·74	40·28	1371
7	6	591	153	90·1	4	688·6	76·76	40·28	1310
8	6	617	162	90·2	5	689·7	76·78	40·28	1388
9	8		163	89·6	5	690·8	76·80	40·27	1420
10	8		168	88·6	4	691·9	76·82	40·27	1471

The ball's diameter, 1·965 inches, and weight 16 oz 12 dr.

The GUN was n° 1, without any of the leaden weights. The gun itself now weighs only 179 lb, as it has been lightened 111 lb, by turning it down, to try if the velocity of the ball would be any less by making the gun lighter: but no difference appears, as the iron work is 205, the gun and iron together this day weigh 384 lb.

80. The PENDULUM as left yesterday, except that it had received a strengthening strap of iron, weighing 7 lb 13½ oz, which, reduced to its centre of gravity, is placed at 79 inches below the axis. With this strap it weighed this morning, before the experiments commenced, 683 lb; which is 6·2 lb less than it ought to be, by adding all the balls and plugs to the first weight; of which 6·2 lb difference, about 1·6 lb is for waste of the leaden facing, and the rest 4·6 lb is probably by evaporation: and as the time the pendulum has hung up is 11 days, the rate of evaporation is about ⅖ of a pound per day. The 6·2 lb loss is divided equally among all the 32 experiments that have been made.

On account of the iron strap of 7·8 lb added at 79 inches, as above, the formula last given, for the variation in the centre of gravity and number of oscillations, will need correction, namely the formula

$$G = 76{\cdot}54 + \frac{i - 76{\cdot}54}{670 + b}b,$$

$$N = 40{\cdot}27 - \frac{i - 86}{217947}bi.$$

Now these formulæ, by making $i = 79$, and $b = 7{\cdot}8$, become

$$G = 76{\cdot}54 + {\cdot}03 = 76{\cdot}57,$$

$$\text{and } N = 40{\cdot}27 + {\cdot}02 = 40{\cdot}29.$$

And hence the corresponding radical values are nearly

p	g	n
678	76·57	40·29

Which values, being substituted in the two general theorems, viz.

$$G = g + \frac{i - g}{p + b}b, \text{ and}$$

$$N = n - \frac{bin(inn - 140850)}{281700pg + bi(inn + 140850)},$$

they become

$$G = 76{\cdot}57 + \frac{i - 76{\cdot}57}{678 + b}b, \text{ and}$$

$$N = 40{\cdot}29 - \frac{40{\cdot}29\, bi(i - 85{\cdot}92)}{8920300 + bi(i + 85{\cdot}92)},$$

$$\text{or } N = 40{\cdot}29 - \frac{i - 85{\cdot}92}{220309}bi \text{ nearly,}$$

which are the new theorems hereafter to be used.

Note, the mean value of i, for the point struck the four last days, is 88·82; which, used in these last formula, give the corrected values of g and n, as inserted in their proper columns in the table of this day's experiments.——N° 7 is doubtful, and therefore omitted.

The means of this day are as below:

Powder wt	Powder ht	Recoil gun	Veloc. of the ball
4	3·0	455	1124
6	4·3	615	1372
8	5·5		1445

81. *Saturday, September 25, 1784.*

This day Major Blomfield alone tried some cartridges, of 8 oz each, by firing them behind, before, and in the middle; but he found no sensible difference in the velocities.

He also discharged several low balls, weighing only 13 oz 3 dr, and having about ·15 of an inch windage; and the same balls, when covered with leather, so as to fit closely in the bore: but the velocities were the same; probably owing to the fired powder quickly blowing off the leather.

The weight of the pendulum was increased 10 or 11 lb, namely, by 8 balls and 58 inches of plugs.

82. *Monday, October 4, 1784; from 11 till 2.*

The weather dry, but cold and windy.

N°	Powder	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. of ball
	oz.			inches	in	lbs	inches		feet
1	4	D 149							
2	4	163							
3	4	164							
4	8	740	158	87·6	6	704·0	77·03	40·26	1440
5	8	742	166	88·2	6	705·0	77·05	40·26	1482
6	8	742	163	88·1	7	706·1	77·07	40·26	1482
7	8	729	158	87·6	6	707·1	77·09	40·25	1425
8	8	749	168	89·0	6	708·2	77·11	40·25	1517
9	8	751	166	88·8	6	709·2	77·13	40·25	1482
10	6	615	151	89·1	6	710·3	77·14	40·25	1367
11	6	604	150	90·2	7	711·3	77·16	40·25	1323
12	6	586	147	92·3	9	712·4	77·18	40·24	1289
13	6	610	152	91·9	8	713·4	77·20	40·24	1321
14	4	457	128	92·5	7	714·5	77·22	40·24	1124
15	4	453	120	92·4	8	715·5	77·24	40·24	1041
16	4	447	120	92·3	6	716·5	77·26	40·24	1059
17	2	270	85	91·5	5	717·6	77·27	40·23	747
18	2	271	85	91·2	4	718·6	77·29	40·23	762
19	2	279	87	91·4	4	719·7	77·31	40·23	768
20	4	459	128	92·3	5	720·7	77·33	40·23	1124

The ball's diameter 1·96, and weight alternately 16 oz 10 dr and 16 oz 14 dr.

The GUN n° 1, without the leads, weighed 384 lb.

The PENDULUM the same as left hanging since the last day.

This day was a continuation of the experiments with the light gun, again to try if the velocity was altered. But without effect. The means as below:

Powder's Weight	2 oz	4 oz	6 oz	8 oz
- - - height	1·73	2·94	4·12	5·42
Recoil of gun	273	454	604	742
Velocity of ball	759	1086	1325	1472

The mean weight of the balls is 16 oz 12 dr.

The mean value of *i*, for the point struck, was 89·3.

33. *Tuesday, October 5, 1784; from 11 till 2.*

The weather fine and warm.

N°	Vibration of gun	Vibration of pend	Point struck	Pigus	Values of *p*	Values of *g*	Values of *n*	Veloc. of ball
			inches	inc	lb	inches		feet
1	48							
2	206	152	89·0	5	721·9	77·35	40·23	1404
3	213	158	87·6	5	723·0	77·37	40·23	1463
4	208	163	91·8	7	724·1	77·39	40·23	1443
5	½	158	91·0	6	725·2	77·41	40·22	1414
6	¾	152	90·7	6	726·4	77·43	40·22	1367
7	¼	153	91·3	6	727·5	77·45	40·22	1374
							mean	1411

The first charge of powder was 4 oz, all the rest 8 oz. The diameter of the balls was 1·96, the first weighed 16 oz 10 dr, all the rest 16 oz 14 dr.

The GUN was n° 1, with 687 lb of lead fixed to it, namely, 433½ lb about the trunnions, and 253½ lb lashed upon the upper side of the gun, close to, and before and behind the stem: these, with 384 lb for the gun and iron together, make in all 1071 lb.

The object was again to try if the velocity of the ball would be increased by diminishing the recoil of the gun. And, for the severer trial, a great quantity of heavy timber was laid behind and against the cascable of the gun in the last three rounds, so as to stop the recoil intirely, which it did, excepting for about the ½ or ¼ of an inch, which the gun pushed the timber back, as expressed in the column of recoil. But the result was still the same.

The PENDULUM the same as left hanging since yesterday.

The mean value of *i*, or point struck the last 6 days, is 89·4.

84. *Wednesday, October* 6, 1784.

The weather clear, but windy.

N°	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of *p*	Values of *g*	Values of *n*	Veloc. of ball
			inches	in	lb	inches		feet
1	62							
2	144							
3	144							
4	148							
5	147							
6	155							
7	157							
8	272	150	88·4	9	728·7	77·46	40·22	1416
9	268	146	88·8	7	729·8	77·48	40·22	1375
10	279	150	88·1	5	730·9	77·49	40·22	1426
11	279	147	88·7	7	732·0	77·51	40·21	1390
12	273	150	87·4	6	733·2	77·52	40·21	1442
13	282	174	87·4	8	693·4	76·83	40·27	1566
							mean	1436

The first charge of powder was 4 oz, all the rest 8 oz. The ball's diameter 1·95, and weight 16 oz 9 dr.

The GUN n° 1, with leads, weighed 817 lb.

The PENDULUM as left yesterday.

The mean value of *i*, or point struck, these last 7 days, 89·3.

The object this day was the effect of cork wads, and of different degrees of ramming. The cork wads were near an inch long, and were made to fit very tight, being rather more than 2 inches diameter; and weighed 5 drams each.

Nos 1, 2, 3, 8, 9 were without wads,
4, 5, 10, 11 with a wad gently pressed home,
6, 7, 12, 13 with a wad, and hard rammed by 2 men.

At n° 12 one of the iron bands of the pendulum broke, and fell across the measuring arch. The band weighed 41 lb; n° 13 was fired after the band was removed, and consequently 41 lb must be deducted.

The velocities are

1416 1375	1396 the mean without wads.
1426 1390	1408 with wads not pressed.
1442 D	1442 with wads very hard rammed.
148	Mean recoil without ball
275	Ditto with ball.

N° 13 is very doubtful, the vibration of the pendulum being evidently too large; perhaps 174 had been set down instead of 164.

In the above there seems to be some small advantage in favour of the wads. But it is suspected the difference is only accidental; and the number of experiments is too small to afford any tolerably good mediums.

85. *Monday, October* 11, 1784; *from* 11 *till* 2.
The weather cold and cloudy.

N°	Ball's weight		Vibration of		Point struck	Plugs	Values of			Veloc. of ball
			gun	pend			*p*	*g*	*n*	
	oz	dr			inches	in.	lb	inches		feet
1			63							
2			143							
3	16	9		141	87·4	10	733·5	77·52	40·21	1357
4	.	9	285	160	85·9	9	734·9	77·54	·	1569
5	.	8	275	150	86·8	7	736·4	77·56	·	1465
6	.	8	267	149	86·1	8	737·8	77·58	·	1470
7	.	8	268	145	86·2	8	739·3	77·60	·	1433
8	.	8	264	146	87·0	9	740·7	77·62	·	1432
9	.	8	274	147	85·4	6	742·2	77·64	·	1472
10	.	8	272	148	86·5	7	743·6	77·66	·	1467
11	.	5	267	142	85·9	5	745·1	77·68	·	1436
12	.	5	261	137	86·5	9	746·5	77·70	·	1379
13	.	5	263	137	85·9	7	748·0	77·72	·	1392
14	.	5	269	143	85·7	6	749·4	77·74	·	1459
									mean	1444

The first charge of powder was 4 oz, all the rest 8 oz; the ball's diameter 1·95.

The GUN n° 1, weighed 817 lb.

The PENDULUM had its band repaired, which did not however alter its weight. The whole weighed this morning 733¼ lb. Now the weight of the balls and plugs in the last 5 days is 62 lb, which, being added to 683 lb, the weight of the pendulum on September 21, it makes 745 lb, which is 11½ lb more than it weighed this morning. For this defect I know of no cause but evaporation: for in this time there was no waste of leaden facing, as the other end of the block was used, which was not covered with lead. The time in which this 11½ lb was lost is 20 days, which is nearly at the rate of ½ a pound each day. This defect is therefore divided equally among all the days.

The mean value of *i*, for the point struck these 8 days, is 88·8.

The object this day was again the effect of cork wads, and different degrees of ramming.

	Ht. of Powder	Mean Veloc.
Nos 2, 9 were without wads - - -	5·85	1472
3, 5, 7 with wads, not rammed - -	5·87	1418
6, 8 a wad, and very hard rammed	4·40	1451
10, 11, 12 a wad, and moderately rammed	5·20	1427
13, 14, 2 wads over powder and 1 over ball, and very hard rammed	4·45	1426
Mean of all - - - - - - - - - - - -	5·15	1444

	Wt. of ball	Mean Veloc.
Nos 3, 4 - - - - - - - - - - - - -	16 9	1463
5, 6, 7, 8, 9, 10 - - - - - -	16 8	1456
11, 12, 13, 14 - - - - - - - - -	16 5	1417
Mean of all - - - - - - - - - - - -	16 7	1444

In this course the wads have no perceptible effect.

86. *Tuesday, October* 12, 1784; *from* 11 *till* 1.

The weather fine and clear.

N°	Vibration of gun	Vibration of pend	Point struck	Plugs	Values of p	Values of g	Values of n	Veloc. of ball
			inches	in.	lb	inches		feet
1	123							
2		208	87·1	9	750·9	77·76	40·21	2047
3	425	219	86·0	9	752·3	77·78	40·21	2187
4	409	200	85·6	10	753·8	77·79	40·20	2011
5		152	87·1	7	755·2	77·81	40·20	1505
6	389	197	85·4	11	756·7	77·83	40·20	1994
7	412	204	85·7	8	758·1	77·85	40·20	2062
8	408	203	85·5	8	759·5	77·86	40·20	2061
							mean	2060

The first charge of powder was 8 oz, all the rest 16 oz; the ball's diameter 1·96 inches, and weight 16 oz 11 dr.

The GUN n° 4, weighed 928 lb.

The PENDULUM as left yesterday. But it was quite broken and useless at the end of these experiments.

The mean value of *i*, or point struck these 9 days, 88·6.

The object this day was the effect of firing the charge in different parts, either before, or behind, or in the middle: for which the means are as below:

	Mean Veloc.
N°s 2, 6 fired before -	2020
3, 7 in the middle -	2124
4, 8 behind - - -	2036
Mean of all -- - - - -	2060
Mean recoil of gun - -	409

N° 5 is omitted as doubtful.

THE EXPERIMENTS OF 1785.

87. Several of the experiments, of the two former years, being not so regular as might be wished, we have again undertaken to repeat some of them, and to add still more to the stock already obtained, that the mediums upon the whole may be tolerably exact, the great number of repetitions counteracting the unavoidable small irregularities, and deviations from the truth, in experiments instituted on so large a scale. For this purpose, we begin with the gun n° 2, and use charges of 8 ounces of powder; and have formed the resolution of firing every shot into a fresh and sound part of the block of wood, and changing the block very frequently, before it become too much battered, that the penetration of the ball and the force of the blow may be obtained with the greater degree of accuracy.

It is also proposed to procure some good ranges, to compare them with the initial velocities made under the same circumstances; from the comparison of which we may estimate the effects of the resistance of the air, and so lay a

foundation for a new theory of gunnery. It is rather difficult to obtain with accuracy such long ranges as our initial velocities would produce, being from 1 mile to 2 miles, when the projection is made at an angle of 45 degrees; for in such long ranges our small balls cannot be seen, when they fall to the ground. We were obliged therefore to have recourse to the water, in which the fall of the ball can be much better perceived; because the plunge of the ball in the water, breaking the surface and throwing it up, makes the place visible at a great distance. But then another difficulty occurs, how to obtain exactly the distance of the fall, or length of the range, as the mark made in the surface of the water is visible but for a moment. This difficulty, however, our situation at Woolwich, close by the river Thames, enabled us to overcome, as well as afforded a good length of range. For, at our situation in the Warren, the river makes a remarkable turn, and forms below us the part called the Gallions Reach, a map of which is here given in plate I. In this map, A denotes the point where the guns were placed, being the Convicts' Wharf, which is so called because it is there that the convicts, or felons condemned to work on the river Thames, land their gravel, and upon which they usually labour. From this point we have a convenient range of about a mile and a half towards B, in the county of Essex, where there is a private or merchant's powder magazine. The buildings near C consist of the academy and a noble range of store-houses; and from this point we should have had a still longer and more convenient range, had not our view from hence been interrupted by four large hulks, which lie, for the use of the convicts, in the river opposite the part between this point and the point A. Having found this convenient situation for our operations, we made an exact survey and map of the two sides of the river, both ways beyond the extent of the ranges; and fixed on convenient stations at D and E on the south side, and F and G on the north side of the river, to place two parties of observers, who might mark the place where

each ball should fall in the water, as well as note down the time employed by the ball in flying through the air, from the visible discharge of the gun to the plunge of the ball in the water. The method of determining the place of the fall was this: Two parties of observers, consisting of three or four steady and intelligent young gentlemen cadets in each party, having taken their stands at D and F, or E and G, according to the expected length of the range, carefully watched the discharge of every ball from the gun at A; then tracing it, as it were, through the air by the loud whizzing noise it made in its flight, their eye was prepared and directed gradually towards the place of the fall, which they seldom missed of observing. Then immediately on perceiving the plunge, some of them noted the time of flight, by a good stop watch, while others observed some remarkable land object on the opposite coast, and directly in a line with the place in the water where the ball fell. This done, they directed the telescope or sights of an instrument, such as a theodolite or plain table, to that object, and noted the position of it. This being done by each party of observers, and the line of position from each station drawn on the plan, afterwards at leisure, the intersection of those lines gave very exactly the place of the fall, and consequently its distance from the gun. In this manner then were determined all the ranges and times of flight registered in the following experiments; those places being left blank where the observation was either doubtful, or not made at all. The times of flight were also sometimes observed at the gun itself, where the plunge of the ball could often be perceived.

In this map of the river in plate I, the dotted line on each side of the river, denotes low-water mark; the first black line next without it denotes high-water mark; and the other, or outermost line, is the land bank which has been raised in former ages, with immense labour, from Greenwich for many miles below, to prevent the waters of the river from overflowing the adjacent fields, which it would do every tide, as they lie low and are otherwise very marshy.

88. *Wednesday, August* 31, 1785.

Employed this day in making part of the survey by the side of the river, for forming the map, and fixing the stations proper for the parties of observers to occupy, in watching the fall of the balls in the river; and for other purposes.

We weighed and measured the pendulum, which had been prepared in a very complete manner, and with stronger bands than before. It weighed just 795 lb. And, by a mean of several times balancing and vibrating, we found $78\frac{1}{3}$ inches to be the distance of the centre of gravity below the axis, and 40·07 the number of oscillations per minute.

After executing part of the survey by the side of the river, we fired a few balls upon the water, from the Convicts' or Proof Wharf, to try whereabouts they would fall, and thereby to judge of the proper places for the observers to be stationed at. The gun was n° 2, with 8 ounces of powder, and was tried at different elevations. When the gun was elevated at 45 degrees, the balls ranged much too far, going beyond the stretch of the river, and falling on the coast of Essex below the point B. But at 15 degrees elevation, the balls ranged to a very convenient distance, namely, a little more than a mile. And their fall in the water could be very well seen from the side of the river nearly opposite the place of the fall, and sometimes from the gun itself.

On this occasion I took out with me, and employed the first class of Gentlemen Cadets belonging to the Royal Military Academy, namely, Messieurs Bartlett, Rowley, De Butts, Bryce, Wm. Fenwick, Pilkington, Edridge, and Watkins, who have gone through the science of fluxions, and have applied it to several important considerations in natural philosophy. Those gentlemen I have voluntarily offered and undertaken to introduce to the practice of these

interesting experiments, with the application of the theory of them, which they have before studied under my care. For, though it be not mv academy duty, I am desirous of doing this for their benefit, and as much as possible to assist the eager and diligent studies of so learned and amiable a class of young gentlemen; who, as well as the whole body of students now in the upper academy, form the best set of young men I ever knew in my life; nay, I did not think it even possible, in our state of society in this country, for such a number of gentlemen to exist together in the constant daily habits of so much regularity and good manners; their behaviour being indeed perfectly exemplary, and the pure effect of true philosophical principles, arising from a rational conviction of the propriety of a regular good conduct, which is such as would do honour to the purest and most perfect state of society that ever existed in the world: and I have no hesitation in predicting the great honour and future services, which will doubtless be rendered to the state by such eminent instances of virtue and abilities.*

89. *Thursday, September* 1, 1785.

Went out with the same class of eight young men, to complete the survey of the river side. The weather changed to rain after we were out, which continued the whole time, and to such a degree as to wet us entirely through all our clothes. Yet every one went through the business, not only willingly, but even chearfully.

* At this distance of time, anno 1812, and long before, the world has had the satisfaction to find, that this prediction has been most amply and accurately fulfilled, in every instance.

90. *Friday, September* 2, 1785; *from* 9 *till* 3.

The weather rather windy and cloudy.

Barometer 29·8; Thermometer within 66°.

Went with Major Blomfield and the same class of cadets, and made the following set of 14 experiments, the first 8 balls being fired into the pendulum, and the other 6 down the river, to get the corresponding ranges.

N°	Vibr. pend	Point struck	Plugs	Values of			Veloc. ball	Penetration
				p	*g*	*n*		
		inches	inc	lb	inches		feet	inches
1	128	81·0	10	795·0	78·33	40·07	1438	18·0
2	139	85·7	9	796·6	35	07	1479	19·9
3	141	90·3	11	798·3	38	07	1428	
4	146	92·7	12	800·0	40	07	1444	16·7
5	163	96·2	12	801·6	43	06	1557	20·3
6	171	94·0	10	803·2	45	06	1675	20·7
7	149	89·1	13	804·8	47	06	1543	16·4
8	144	91·5	11	806·4	50	06	1457	16·6
						means	1503	18·9

	Time sec	Range feet
9	14	6110
10		6060
11		not
12		seen
13		5760
14		5735
	mean	5916

The charge of powder was always 8 oz; the ball's diameter 1·965, and weight 16 oz 13 dr.

The GUN was n° 2. It was not hung on an axis, as in the two former years, but mounted on a small carronade carriage, made for the purpose, both in the last 6 rounds, which were fired down the river, and in the first 8 rounds, which were fired into the new pendulum, at the same distance as formerly, or about 35 feet, and each ball into a

fresh part of the wood, both to obtain the force of the blow the more accurately, and to take the penetration of the ball in the solid wood, which we did every time by pushing in a wire to touch the hinder part of the ball: these penetrations are various, according as the part struck was more or less compact, and they are rather larger than was expected, the medium of all being $18\frac{4}{10}$, though the block of elm, as the carpenters assured us, was sound, dry, and well-seasoned wood. The penetrations are set down in the last column, and are for the fore part of the balls, the diameter having been always added to the length of the wire.

The POWDER was not of the same parcel as the two former years; but it was from the same maker, and made as nearly similar to the former as might be. The charges were gently set home, and all circumstances made alike. The mean length of the charge of 8 oz was 4·84.

The PENDULUM had been kept close covered with a painted canvas cloth since the first day that it was weighed and measured, to preserve it from the weather. The plugs weighed 9 ounces to every 11 inches in length; the whole weight of all the plugs, together with that of the 8 balls, make up 13 lb, wanting only an ounce and a half; and when the pendulum was taken down and weighed this afternoon, its weight was found to be 808 lb, which is just 13 lb more than its weight at first. So that it has neither lost weight by evaporation, nor gained by imbibing moisture: owing, probably, to the circumstance of being covered by the painted canvas. All the apparatus was in good order, and the experiments all very accurately made.

At the beginning of these experiments, the values of p, g, n, being $p = 795, g = 78\frac{1}{3}, n = 40{\cdot}07$; if these values be substituted in the two theorems

$$G = g + \frac{i - g}{p + b} b,$$

$$N = n - \frac{bin(inn - 140850)}{281700\,pg + bi(inn + 140850)},$$

for the correction of g and n, they become

$$G = 78{\cdot}33 + \frac{i-78{\cdot}33}{p+b}b,$$

$$N = 40{\cdot}07 - \frac{40{\cdot}07\,bi(i-87{\cdot}16)}{10855700 + bi(i+87{\cdot}16)},$$

$$\text{or } N = 40{\cdot}07 - \frac{i-87{\cdot}16}{270940}bi \text{ nearly.}$$

And by these theorems the numbers in the columns g and n are made out, the mean value of i, or point struck, being 90·1.

The last 6 rounds were fired down the river from the Convicts' or Proof Wharf at A, and the place of the fall observed by two parties of the cadets, stationed at D and E. The gun had 15 degrees elevation. The fall of the first only could be seen at the gun, where the time of flight was observed by a stop watch, and found to be 14 seconds. The two parties of observers at D and E had no time-piece with them, so that the other times of flight could not be observed. The medium range is 5916 feet or 1972 yards. The last two balls went close over the heads, and just fell beyond, the lower party of observers, at E; yet notwithstanding their imminent danger, they gallantly resolved to keep their ground, if any more rounds should be fired, not knowing immediately that we intended not firing any more at that time. These two rounds were probably deflected thus a little from their course by the usual causes of deviation. And perhaps the two former rounds had been still farther deflected, and thrown on the land, as the observers saw nothing of them. But the gun was pointed in a direction rather nearer this south side of the river.

91. *Thursday, September* 8, 1785; *from* 12 *to* 3.

The weather close and warm, rather hazy.

Barometer 30·02; Thermometer 65° within, but warmer without.

N°	Powder	Ball's wt		Ball's diam	Time	Range	Whereabouts the Balls fell
	oz	oz	dr	inches	sec	feet	
1	8	16	12	1·96	14	6460	Near the middle of the river
2	.	.	.	.		6080	Near the north side
3	.	.	.	.			
4	.	.	.	.	15	6040	Ditto
5	.	.	.	.	15½	6540	Ditto
6	.	.	.	.	15	6460	Near the middle
7	.	.	.	.	14	5720	On the south bank, and within 40 yds. of the lower station E
				means	14·7	6216	

These 7 rounds were fired down the river from the same place as before; the elevation of the gun being 15 degrees, and all other circumstances the same as before. The gun was pointed nearly to the middle of the river; yet the balls fell mostly wide of the direction, and that both ways, some falling near one side of the river, and some near the other, though there was not the least wind. The times of flight were taken with a stop watch, at the lower station of observers at E, by noting the time between seeing the flash of the gun and the plunge of the ball in the water. They run from 14 to 15½ seconds, and accord very well with the ranges, the larger to the larger: the medium is 14·7 seconds; and the medium range 6216 feet, or 2072 yards. N° 3 was not seen. The mean length of charge 4·8 inches.

The same parties of young gentlemen kept their station very gallantly, and made no hesitation in offering to attend and observe there for the remainder of the experiments, though some of the balls this day again fell near them, and one indeed within 40 yards of them.

92. *Friday, September* 9, 1785; *from* 9½ *till* 1.

The weather very fine and warm.

Barometer 29·93; Thermometer, within, 66°.

N°	Vibrat pend	Point struck	Plugs	Values of			Veloc. ball	Penetr.
				p	*g*	*n*		
		inches	inches	lb	inches		feet	inches
1	99	79·0	9	805·3	78·48	40·06	1162	16·7
2	107	82·3	10	806·7	49	06	1208	16·4
3	114	87·1	10	808·1	50	07	1218	16·7
4	115	87·3	9	809·6	51	07	1228	
						mediums	1204	16·6

The charge of powder was 4oz; the ball's diameter 1·965, and weight 16 oz 12 dr.

These 4 rounds were fired into the same pendulum as was left hanging on September 2, which had been kept under cover since that time. After these 4 rounds, it weighed 811 lb, which is 2¾ lb less than it ought to be, when the weight of the 4 balls and plugs are added to its former weight, and which 2¾ lb it must be supposed to have lost by evaporation in the course of the 7 days, which was mostly dry, warm weather.

The plugs weighed 9 oz to 14½ inches.

Mean length of the charge 3·0.

We could not venture to fire down the river this day, on account of the great number of ships that were upon it.

93. *Saturday, September* 10, 1785; *from* 12 *till* 2.

Fine dry weather.

Barometer 29·8; Thermometer 66°.

N°	Powder	Ball's wt		Ball's diam	Penetr.	Recoil
	oz	oz	dr	inches	inches	inches
1	2	16	10	1·96	6·1	2·5
2	4	.	.	.	12·2	7·0
3	8	.	.	.	20·8	15·8
4	2	.	.	.	6·7	3·0
5	4	.	.	.	14·4	9·0
6	8	.	.	.	23·0	17·5
7	2	16	12	.	7·8	2·5
8	4	.	.	.	14·0	7·5
9	8	.	.	.	20·7	

These 9 balls were fired into the root end of a block of elm, laid upon the ground, to obtain the penetration with different charges, each ball being fired into a fresh and sound part of the wood, and in the direction of the fibres. The wood was moist within, as was discovered by boring out the balls; but it was hard and firm of its kind, being in the root, or the root end after the body of the tree was sawed off from it. The penetrations are for the fore part of the ball, as usual.

The gun was n° 2, and mounted, as in all the experiments of this year, on a small sea gun carriage, without trucks, but fixed on a base like a mortar bed, and slid along the ground or platform in recoiling.

The muzzle was placed at 79 inches from the face of the block. The mean penetrations and recoils are as follow:

Powder	Penetr.	Recoil
2 oz	6·9	2·7
4	13·5	7·8
8	21·2	16·7

So that the penetrations are nearly 7, 14, 21, or nearly as 1, 2, 3, or as the logarithms of the weights of powder.

94. *Wednesday, September* 14, 1785; *from* 10 *till* 12½.

A fine warm day.

Barometer 30·5; Thermometer, within, 67°.

N°	Powder	Ball's wt		Ball's diam	Time	Range
	oz	oz	dr	inches	sec	feet
1	4	16	12	1·96		
2	.	.	.	.		
3	.	.	.	.	10	4730
4	.	.	.	.	7¼	4030
5	.	.	.	.	8	4450
6	.	.	.	.		4380
7	.	.	.	.		
8	.	.	.	.		
				mediums	8·4	4398

These 8 rounds were fired down the river as before. The gun n° 2, and elevation 15 degrees, as usual. One party of the young gentlemen was stationed at D as before, but the other on the north side of the river at Duval's house at F. This last party saw only one ball plunge, and the first party saw four; which however proved sufficient for determining their ranges, because they all fell near the middle of the river, a circumstance which we also at the gun could sometimes perceive.

The mean time of flight was about 8¼ seconds, and the mean range 4398 feet, or 1466 yards.

95. *Saturday, September* 17, 1785.

N°	Powder	Ball's wt		Ball's diam	Penetr
	oz	oz	dr	inches	inches
1	12	16	12	1·96	22·0
2	14	.	.	.	23·6
3	16	.	.	.	24·0
4	10	16	14	1·97	22·3
5	8	.	.	.	18·1

These 5 balls were fired into the same block of elm root as on the 10th instant, to get a greater variety of penetrations.

96. *Tuesday, September* 27, 1785.

N°	Powder	Ball's wt		Ball's diam	Penetr.	Part of the Charge fired at
	oz	oz	dr			
1	8	17	0		20·5	Back part
2	.	.	.		20·6	
3	.	.	.		21·6	Middle
4	.	.	.		20·5	
5	.	.	.		11·0	Fore part
6	.	.	.		17·3	

These 6 also were fired, from the same gun, into the same block, to try the difference by firing the cartridge either behind, or before, or in the middle.——There must be some mistake in the numbers in the last two rounds, which cannot possibly differ so much from the other numbers.

97. *Wednesday, September* 28, 1785.

A fine clear day.

Barometer 30·35; Thermometer 60.

N°	Powder	Ball's wt		Ball's diam	Elevation	Time	Range
	oz	oz	dr	inches	degrees	sec	feet
1	4	16	10	1·97	15		5180
2	4	•	•	•	•	8¾	4370
3	4	•	•	•	•		
4	4	16	7	•	•	7¾	4020
5	2	•	•	•	45		5120
6	2	•	•	•	•	21½	5300
7	2	•	•	•	•	21	5200
8	2	•	•	•	•		4120
9	4	16	11	1·95	15		
10	4	•	•	•	•	13½	5770
11	2	1 6	7	1·97	45	23	5600
12	2	16	10	•	•		

These 12 rounds were fired down the river: the gun, stations, parties of cadets, &c, as before. The fall of those balls was not seen whose range is not set down. With 2 oz. of powder the gun was elevated 45 degrees, but with 4 oz only 15 degrees, as before. The mediums are as below:

Powder	Elev.	Time	Range
2 oz	45°	22″	5068
4	15	8¼	4523

Rejecting n° 10, as very doubtful; a mistake most likely having been made in the weight of the powder.

98. *Thursday, September* 29, 1785.

A fine clear day.

Barometer 30·35 ; Thermometer 60.

N°	Powder	Ball's wt		Ball's diam	Elevation	Time	Range
	oz	oz	dr	inches	degrees	sec	feet
1	2	16	12	1·95	45	20	5120
2	.	.	.	.	.	19	4730
3	.	.	.	.	.	20½	5370
4	.	.	.	.	.	20	5120
5		.	.	.	.	22½	5510
6		.	.	.	.	20	5050
7	12	.	.	.	15	17	7120
8	.	.	.	.	.	10 D	4860 D
9	.	.	.	.	.	9 D	4880 D
10	.	.	.	.	.	14	6660
11	.	.	.	.	.		5500
12	.	.	.	.	.		7520

These 12 rounds were fired on the river, and observed as before. N°s 8 and 9 are very doubtful: the means of the rest are as below:

Powder	Elevation	Time	Range
2 oz	45°	20⅓	5150
12	15	15½	6700

99. The same day the following 6 rounds were fired into the block of elm root, to try the penetrations with and without wads; the first 4 being with a wad over the powder, and hard rammed; the other two without.

N°	Powder	Ball's wt		Ball's diam	Penetration	
	oz	oz	dr	inches	inches	
1	8	15	12	1·95	16·1	With wads
2	.	.	.	.	21·4	
3	.	.	.	.	20·6	
4	.	.	.	.	19·8	
5	.	.	.	.	19·8	Without wads
6	.	.	.	.	21·0	

100. *Tuesday, October* 4, 1785.

Fine morning, but turned to rain about noon.

Barometer 29·93.

N°	Powder	Ball's wt		Ball's diam	Time	Range
	oz	oz	dr	inches	sec	feet
1	8	15	3	1·96		6330
2	.	.	.	.		5770
3	.	.	.	.		
4	.	.	.	.	$8\frac{1}{2}$	4800
5	.	.	.	.		4880
					medium	5600

These 5 rounds were fired on the river, and observed as before.

The Gun was n° 3, and its elevation 15 degrees.

The balls were not good ones, and the ranges very irregular; and the medium 5600 feet, or 1867 yards, too low; perhaps owing to the lightness of the balls.

101. *Tuesday, October* 11, 1785.

The weather fine.

Barometer 29·88; Thermometer 60.

N°	Powder	Ball's wt		Ball's diam	Time	Range
	oz	oz	dr	inches	sec	feet
1	8	15	12	1·96		5580
2	.	.	.	.		
3	.	.	.	.	$10\frac{1}{4}$	5270
4	.	.	.	.		5990
5	.	.	.	.	9	4910
6	.	.	.	.	11	5750
7	.	.	.	.	$9\frac{1}{2}$	6140
8	.	.	.	.	11	5700
means					$10\frac{1}{7}$	5620

These 8 were fired in the river, and observed as before.

The Gun was n° 3, and was elevated 15 degrees.

The ranges are again low, probably from the lightness of the balls.

The usual causes of deflection carried three of the balls, namely, the 1st, 7th, and 8th, very near the south station at E; and then fell almost close to the party there. In general it was observed that the balls deviate from their line of direction, or middle line of the river, to each side, by half the breadth of the river, or from 300 to 400 yards!

EXPERIMENT IN 1786.

102. *Monday, June* 12, 1786; *from* 10 *till* 1.

Fine weather.

Barometer 29·89; Thermometer 63° at 9 A. M.

The GUN was n° 2.

N°	Powder	Ball's diam	Ball's wt.		Elevat gun	Time of flight	Range
	oz	inches	oz	dr	deg.		
1	2	1·96	16	6	15		
2	.	1·96	.	6	.	14 D	5000 D
3	.	1·96	.	6	.		5040 D
4	.	1·96	.	6	.	8	3920
5	.	1·97	.	7	.	7½	3560
6	.	1·96	.	5	.	8½	3910
7	.	1·96	.	5	.	10½	4450
8	.	1·96	.	5	.	9½	4280
9	.	1·95	.	5	.	8½	3910
10	.	1·96	.	4	.	15 D	5600 D
11	.	1·96	.	4	.	8½	3910
12	.	1·96	.	4	.	11½	4750
13	.	1·96	.	4	.	9½	4270
14	.	1·96	.	4	.	10	4230
15	.	1·96	.	4	.	9½	4000
16	.	1·95	.	3	.		4960 D
17	.	1·95	.	9	.	9½	4420
18	4	1·95	.	3	.		4840
19	4	1·96	.	3	.	11½	4690
20	4	1·96	.	2	.	10½	5650
mediums	2	1·959	16	5	15	9¼	4130
	4	1·957	16	2⅔	15	11	5060

The ranges were taken from observations, as before, at Duval's house, and the first gibbet. The first 17 rounds were fired this year, with two ounces of powder, to complete the series of ranges at 15 degrees elevation of the gun; and the last three rounds, with 4 ounces, to try if the powder was of the same strength as before: and which, by comparing these three ranges with those of last year, ap-

pears to be now somewhat stronger. So that these ranges and times, it may be presumed, are too great in respect of those of last year. They are also evidently very irregular; owing perhaps to the inequalities of the balls, which were only the remaining outcasts from the whole stock we first began with, having been rejected either from their lightness, or from the irregularities of their surfaces. And sometimes indeed the ranges and times, here set down, were not very accurately obtained. The mediums of all, except those marked doubtful, are placed at the bottom.

A SUMMARY OF THE EXPERIMENTS; WITH PHILOSOPHICAL REMARKS AND DEDUCTIONS.

103. We have now got so far through this long annual course of experiments; and have detailed them in so minute and circumstantial a manner, as to enable every person fully to comprehend and make his own use of them; without subjecting him to the dissatisfaction of having mediums and results forced upon him, unaccompanied with the fair and regular means of assuring himself both of their justice and propriety. We are now therefore to make some use of these experiments, by pointing out the philosophical laws and deductions that flow from them, and making such other remarks as may be suggested by the various circumstances attending of them, or that may be useful for improving or further extending experiments, accompanied with such important consequences in natural philosophy. And for these beneficial purposes, it will first be necessary to bring the mediums and results together into an abstract, or one comprehensive point of view; to form as it were the sure and regular foundation for the future structure we hope to be able to raise upon them.

OF THE LENGTH OF THE CHARGE.

104. And first we shall deduce the lengths or heights of the charge of powder, for every 2 ounces in weight; or the

part of the bore of the gun which every charge occupies: a thing very necessary both to show the part of the bore, occupied by the charge, corresponding to the greatest or any other velocity of the ball, as also to compute *a priori*, from theory, the velocity due to every charge of powder. Now the length of the charge was taken at every experiment, by means of the divisions of inches and tenths marked on the rammer, the mediums of most of them being specified for each day in the preceding account of the experiments; and those mediums of each day are here in the following table collected and ranged in columns, each under its respective weight at top, extending from 2 to 20 ounces:

2	4	6	8	10	12	14	16	18	20
1·7	3·2	4·6	5·9	6·93	8·4	9·8	10·6	12·3	13·2
1·7	3·2	4·4	5·9	6·97	8·2	9·27	11·0	12·15	13·3
2·1	3·3	4·5	6·1	6·9	8·3	9·55	11·2		13·2
1·8	3·23	4·43	6·2	6·97	8·23	9·4	11·4		
1·8	3·33	4·2	5·67	7·0	8·3	9·53	10·9		
1·7	3·24	4·5	6·0		8·37		11·0		
1·7	3·2	4·44	5·7		8·07		10·8		
1·87	3·2	4·4	5·72				11·13		
1·9	3·1	4·37	5·6				11·38		
1·85	3·17	4·27	5·63				11·26		
1·9	2·9	4·28	5·65				11·1		
1·8	3·4	4·12	5·6				10·6		
1·9	3·13		5·6				10·97		
1·9	3·1		5·83				10·87		
1·83	3·08		5·7				10·85		
1·73	3·4		5·77				10·79		
	3·1		5·88						
	3·0		5·45						
	2·95		5·4						
	3·0		5·74						
	3·1		5·85						
	3·1		5·4						
	3·03		4·84						
			4·8						
1·82	3·15	4·38	5·66	6·95	8·27	9·51	10·99	12·22	13·23

and in the lowest line are set down the means among all the former mediums, or numbers in each column, the numbers

in which last line of means are found by adding into one sum the numbers in each column, and dividing that sum by the number of those parts. And thus we have obtained the mediums of the mediums for each day, which must be very near the truth. But to find how near they are to the truth, and to correct them, let these be collected and ranged as in the second column of the following table of the heights of

Wt. oz	Irregular		Regular		Correct means
	means	diffs	diffs	means	
2	1·82			1·85	2·18
		1·33	1·27		
4	3·15			3·12	3·45
		1·23	1·27		
6	4·38			4·39	4·72
		1·28	1·27		
8	5·66			5·66	5·99
		1·29	1·27		
10	6·95			6·93	7·26
		1·32	1·27		
12	8·27			8·20	8·53
		1·24	1·27		
14	9·51			9·47	9·80
		1·48	1·27		
16	10·99			10·74	11·07
		1·23	1·27		
18	12·22			12·01	12·34
		1·01	1·27		
20	13·23			13·28	13·61

charges, or column of irregular means. Then take the differences between each of these, and place them in the 3d column, or irregular differences; which would have been all equal if the mediums had been regular. Find then a medium among these unequal differences, by dividing their sum by the number of them, and it will be found to be 1·27, which set in the 4th column of regular or equal differences. Then, as the numbers in the 3d column, the nearest to this mean 1·27, are the differences between 6, 8, and 10 ounces, by supposing 5·66 to be the true length of the 8 oz charge, all the others are formed from it, by adding and subtracting continually the mean or common difference 1·27, and are placed in the 5th column; which will therefore consist of the true regular length of each charge, including both the powder and the neck of the flannel bag which contained it, as indicated by the marked divisions on the rammer by which the charge was pushed home in the gun bore.

But an addition of 0·33 must be made to each of the numbers in this 5th column, for the length of cylinder of equal capacity with the concave hollow at the bottom of the bore of each gun. Now this concavity was the half of an oblate spheroid, formed by an ellipse whose transverse axis was 2 inches, the diameter of the bore, and its conjugate 1 inch; the depth of the cavity was therefore ·5, the semi-conjugate; and the depth of an equal cylinder nearly ·33. This number then being added to those in the 5th column, will give the n^{os} in the 6th or last column, being the length of cylinder occupied by the powder and the bag together.

How much of each space was really occupied by the bags, may be thus found: the first number 1·85 is the length of the charge of 2 ounces, including the neck; and the common difference 1·27 is the real length of 2 ounces of powder in the bore; therefore, subtracting this number from the former, the remainder 0·58 is the mean length of the bore which was occupied by the neck and bottom of the bag in every charge. And therefore, taking this number from each of those in the last column, the remainders will show the real length of bore occupied by the powder alone in each of the charges.

OF THE RECOIL WITHOUT BALLS.

105. Next let us consider the quantity of recoil, or extent of the vibration of each gun, for every charge of powder; and first without balls. Now as these recoils were measured sometimes to one radius, and sometimes to another, it will be proper to reduce them all to a common radius, as well as to a common weight of gun when it happens to vary in weight. In the first year's experiments, the radius was various, and the chords of recoil were always taken in inches; but in those of the second and third years,

the radius was constantly 10 feet, or 120 inches, which was divided into 1000 equal parts, and the chords of vibration measured in thousandth parts of the radius, each part being $\frac{12}{100}$ of an inch. It will therefore be convenient to reduce the recoils of the first year, to the same radius and parts as those of the other two years: which may be done as follows:

Let r = any radius of the first year in inches,
and c = a corresponding chord of recoil, in inches and parts.

Then $r : 120 :: c : \frac{120c}{r}$ the chord corresponding to the radius 120, and measured in inches;

and $120 : 1000 :: \frac{120c}{r} : \frac{1000c}{r}$ the same chord as expressed in thousandth parts of 120 inches.

Hence then, to reduce any chord of recoil, in the first year, multiply it by 1000, and divide the product by its own radius in inches; so shall the quotient bet he corresponding chord answering to the radius 120 inches, and expressed in thousandth parts of that radius.

106. By the foregoing rule then having reduced all the chords of recoil to the radius 10 feet, and denoted them in thousandth parts of that radius; the mediums of every day's experiments, collected and arranged, are as follow.

Tables of Recoil without Balls.

Charge of Powder oz	2	4	6	8	12	16
	22	53	85	113	176	221
	21	53		119	165	215
	21	55		116		220
GUN n° 1	23	54		110		
	23	55		108		
	22			128		
	22			127		
mediums	22	54	85	117	171	219
	23	52		123		236
	23	56		118		240
GUN n° 2	24			124		
	23			124		
	24					
mediums	23	54		122		238
	22	57	93	123		247
	23	59		125		250
	23	58		125		259
	23	56				252
GUN n° 3	23					
	23					
	24					
	25					
mediums	23	57½	93	125		252
	25	58		127		280
	24	58		131		255
GUN n° 4	26	56				261
	24	59				
mediums	25	58		129		265

Some of these mediums have not the greatest degree of exactness that they are capable of, for want of a sufficient number of repetitions, or numbers to take the mediums of. However, by a very small and obvious correction, the more accurate mediums, for the most usual charges of 2, 4, 8, and 16 ounces of powder, may be fairly stated as follows:

Gun	Centre of gravity	Vibrat.	Axis of gun below axis of recoil	Length of bore	Powder			
					2 oz	4 oz	8 oz	16 oz
n°	inches			inches	Recoils without Balls			
1	80·47	40·1	89·06	28·5	22	53	117	220
2	80·47	40·0	89·06	38·4	23	55	121	237
3	80·50	39·9	89·19	57·7	24	57	125	252
4	80·44	39·8	89·28	80·2	25	59	129	265

The recoils being estimated in parts of which the radius is 1000: and the common weight of the gun, with its frame and leaden weights, being 917 lb; also the distance of the centre of gravity below the axis, and the number of vibrations per minute, as set down in the 2d and 3d columns of the tablet above.

107. From the view and consideration of these numbers, various observations easily arise. As first, that, by observing the four columns, or vertical rows, it appears that the recoil of the gun, and consequently the force of the powder upon it, always increases as the length of the gun increases, and that in a manner tolerably regular as far as the charge of 8 ounces; but after that, the increase is faster: thus, between the shortest bore of 28 inches long, and the longest of 80 inches, the increase in the velocity of recoil, with 2 ounces of powder, is from 22 to 25, or about the $\frac{1}{9}$ part; with 4 ounces of powder, it is from 53 to 59, or the $\frac{1}{9}$ part; and with 8 ounces of powder, it is from 117 to 129, or the $\frac{1}{10}$ part; but with 16 ounces of powder, the increase is from 220 to 265, or the $\frac{1}{8}$ part. And this increase of recoil is chiefly, if not intirely, to be ascribed to the longer time the fluid of the inflamed powder acts upon the gun, in passing through the greater length of bore; at least as far as to the charge of 8 ounces: but the extraordinary increase in the case of 16 ounces, seems to be partly owing to that, and partly to some of the powder, in this high charge, being blown out unfired from the short gun. And from this circumstance we may infer, that the whole of the charge of

8 ounces, without ball, is fired before it issues from the mouth of the short gun, that is before the fluid expands through a space of $22\frac{1}{2}$ inches of bore. And hence, if the velocity of the fluid were known, we could assign the time within which all the powder is fired. If, for instance, the mean velocity of the fluid were only 5000 feet in a second, though perhaps it may be more, the time would be only about the 250th part of a second in which the 8 ounces would be all inflamed.

The foregoing are the rates of increase in the chord of recoil, or in the velocity of the gun, which is proportional to it. It must be remarked, however, that the increase in the *force* of the powder will be about double to that of the recoil, because the force is as the square of the velocity: so that, from the shortest gun to the longest, the increase in the force of the powder, with 2, 4, or 8 ounces, is about $\frac{1}{4}$, or from 4 to 5; and with 16 ounces of powder, the force is almost as 2 to 3, or the increase almost one half of the less force.

108. Again, if we contemplate the numbers on each horizontal line, that is, the recoils of each gun separately, with the several charges of 2, 4, 8, and 16 ounces of powder, we shall find that, in each of them, the recoil increases from the beginning, to a certain part, in a greater ratio than the constant ratio, 2 to 1, of the powder increases; and afterwards in a less ratio than that of the powder. That the ratio of the recoils, in every gun, is greatest at first, or with the least charges of powder: that the ratio continually decreases as the charge increases: that the ratio, at first, is greatest with the shortest gun, and so gradually less and less all the way to the longest: but that, however, the ratio in the shorter guns decreases faster than in the longer; and so as to come sooner to the ratio of 2 to 1 in the shorter guns, than in the longer; and after that, the ratios in the short guns, with the same charge, are less than in the long ones. All these properties will perhaps appear still plainer by arranging together the several ratios for each n° of gun, as here below:

Powder	Ratios for the Gun			
	n°. 1	n° 2	n° 3	n° 4
2				
4	2·41	2·39	2·37½	2·36
8	2·21	2·20	2·19	2·18
16	1·88	1·96	2·02	2·06
means	2·17	2·18	2·19	2·20

where each column of ratios is found by dividing the recoils successively by each other, from the beginning, namely, the recoil of 4 oz by that of 2, the recoil of 8 oz by that of 4, and the recoil of 16 oz by that of 8. Also the first and second lines rather decrease, but the 3d rather increases, and the last, or that of means, also rather increases.

And if we divide the first ratios, in the last table but one, successively by each other, the 2d by the 1st, and the 3d by the 2d; and then again these last ratios or quotients by each other; and so on; we shall obtain the several orders of ratios for each gun, as follows, observing uniform laws:

N° 1			N° 2			N° 3		
22			23			24		
	2·41			2·39			2·37½	
53		·917	55		·920	57		·922
	2·21			2·20			2·19	
117		·850	121		·891	125		·922
	1·88			1·96			2·02	
220			237			252		

N° 4			
25			
	2·36		
59		·924	
	2·18		1·02
129		·945	
	2·06		
265			

where the first column, of each n° or gun, contains the recoils with 2, 4, 8, 16 ounces of powder; the 2d the first

ratios, or the ratios of the recoils; the 3d contains the 2d ratios, or the ratios of the first ratios; and the last column contains the 3d ratios, or the ratios of the 2d ratios.

Or, perhaps, for some purposes it will serve better to set the same table in the following form, where the vertical columns are changed into horizontal lines:

No 1	No 2	No 3
22 53 117 220	23 55 121 237	24 57 125 252
2·41 2·21 1·89	2·39 2·20 1·96	2·37½ 2·19 2·02
·917 ·850	·920 ·891	·922 ·922
·93	·97	1·00

No 4

25 59 129 265
2·36 2·18 2·06
·924 ·945
1·02

OF THE RECOIL WITH BALLS.

109. By collecting now the mean recoil of each gun for every day, after reducing them all to the same weight of gun, 917 lb, and weight of ball, 16 oz 13 dr, and to the same radius 1000, in the manner specified in Art. 105, they will stand as in this following table.

Powder, oz	2	4	6	8	10	12	14	16
Gun n° 1	90	148	197	241	260	290	294	329
	91	145	196	226	253	273	295	331
			199	234	260	281	313	331
				232		287		
				234				
				240				
				244				
				239				
mediums	90	146	197	236	258	283	301	330
Gun n° 2	92	152	207	249	274	296	305	360
	95	157		244	276	304	343	364
				244				348
				246				
mediums	94	154	207	246	275	300	324	358
Gun n° 3	99	166	216	259				399
	100	163	218	257				380
		164		260				
mediums	99	164	217	259				390
Gun n° 4	101	163		266				397
								417
mediums	101	163		266				407

Some of these mediums are not very accurate, for want of a good number of repetitions, and especially those of the last gun n° 4, which has only one duplicate. In this gun, the recoils appear to be all rather low, in respect of the others, but more especially that with the charge of 4 oz of powder, which is evidently much more defective than the rest, and requires an increase of about 6 to make it uniform with the others, and which increase it would probably have received from future experiments, had there been any repetitions of it. Augmenting therefore only that number by 6, all the orders of means will be tolerably regular, and stand as below, for the most usual charges of powder, namely, 2, 4, 8, and 16 ounces.

Gun n°	Powder			
	2	4	8	16
	Recoils with Balls			
1	90	146	236	330
2	94	154	246	·358
3	99	164	259	390
4	101	169	266	407

The common weight of ball being 16 oz 13 dr, and the weight of the gun 917 lb; the other circumstances being as in Art. 106.

110. From the several vertical columns of this tablet of means, we discover, that the recoils increase always as the length of the gun increases; but that in the 4th or longest gun, the increase is less, in proportion, than in the others. And from the horizontal lines we perceive, that the recoil always increases as the charge of powder increases, and that in a manner tolerably regular; and also in continued geometrical proportion when the charges of powder are so; but the common ratio in the former progression being only about $\frac{3}{4}$ of that in the latter. For, if the mediums, for each gun, be divided by each other, namely, the 2d by the 1st, the 3d by the 2d, and the 4th by the 3d, the quotients or ratios will come out as in the following tablet:

Powder	Ratios for the Gun			
	n° 1	n° 2	n° 3	n° 4
2				
4	1·62	1·64	1·66	1·67
8	1·61	1·60	1·58	1·57
16	1·40	1·45	1·50	1·53
means	1·54	1·56	1·58	1·59

where the numbers in the vertical columns, or the ratios for each gun separately, continually decrease; and the numbers in the horizontal lines, or for the different guns with the same weights of powder, rather increase in the first and

third line, but decrease in the second, and again rather increase in the last, which are the mediums of the three ratios in each column, and which mean ratios are rather more than $\frac{3}{4}$ of 2, the common ratio of the weights of powder, which are 2, 4, 8, 16 ounces.

And if we divide the numbers or ratios, in each column, continually by each other; and their quotients by each other again; the whole continued series or columns of ratios, for each gun, will be as here below:

N° 1	N 2	N° 3
90	94	99
1·62	1·64	1·66
146	154	164
·994	·976	·952
1·61	1·60	1·58
·88	·93	1·00
236	246	259
·870	·906	·950
1·40	1·45	1·50
330	358	390

N° 4

101
1·67
169
·940
1·57
1·04
266
·974
1·53
407

where the first column, in each n° or gun, contains the recoils with 2, 4, 8, 16 ounces of powder; the 2d column contains the ratios of those recoils; the 3d contains the 2d ratios, and the last the 3d ratios.

Or the same table may, for some purposes, be more conveniently placed as below, where the vertical columns are ranged in horizontal lines:

N° 1	N° 2	N° 3
90 146 236 330	94 154 246 358	99 164 259 390
1·62 1·61 1·40	1·64 1·60 1·45	1·66 1·58 1·50
·994 ·870	·976 ·906	·952 ·950
·88	·93	1·00

N° 4

101 169 266 407
1·67 1·57 1·53
·940 ·974
1·04

OF THE MEAN VELOCITY OF THE BALL FROM THE RECOIL OF THE GUN.

111. Having determined the mean recoil of the guns, both with and without balls, for the charges of 2, 4, 8, 16 ounces; we can now assign the mean velocity of the ball, for each gun and charge, from the recoils; if, as Mr. Robins has asserted, the force of the powder upon the gun be the same, whether it is fired with a ball or without one. For, if that property be generally true, then the velocity of the ball must be proportional to the difference of the chords of recoil with and without a ball; and that difference being multiplied by a certain constant number, will give the velocity of the ball itself; as we have before shown.

Now if c denote the difference of those chords, b the weight of the ball, G the weight of the gun, g the distance to its centre of gravity, i the distance to the axis of the bore, and n the number of oscillations the gun would make in a minute; then we have found, in Art. 68, that $\frac{59}{96} \times \frac{Ggc}{bin}$ will express the velocity of the ball. And that when $G = 917$, $g = 80{\cdot}47$, $i \doteq 89{\cdot}15$, and $n = 40$, which are the medium values of those letters, then the same theorem becomes $\frac{51}{4} \times \frac{c}{b}$ for the velocity of the ball. And further, when the mean value of b is 1·051 or 16 oz 13 dr, the same theorem for the velocity becomes barely $12\frac{1}{7}c$. Subtracting however the 700th part in the gun n° 1, and adding in the other three guns, as follows, namely,

the 1000th part in n° 2,
400th part in n° 3,
300th part in n° 4.

Therefore, if each of the recoils without balls, in the last table of Art. 106, be taken from the corresponding recoils in

Art. 109, and the remainders be multiplied by $12\frac{1}{7}$, making the additions and subtractions above-mentioned, we shall have the corresponding velocities of the ball by this method. And a synopsis of the whole, for each gun and charge, will be as in the following table:

Charges,	2 oz				4 oz			
Gun n°	Recoil with ball	Recoil without ball	Diff.	Velocity of the ball	Recoil with ball	Recoil without ball	Diff.	Velocity of the ball
1	90	22	68	825	146	53	93	1127
2	94	23	71	863	154	55	99	1203
3	99	24	75	913	164	57	107	1302
4	101	25	76	926	169	59	110	1340
Charges,	8 oz				16 oz			
1	236	117	119	1443	330	220	110	1334
2	246	121	125	1520	358	237	121	1471
3	259	125	134	1631	390	252	138	1680
4	266	129	137	1669	407	265	142	1730

And we shall hereafter see how far these agree with the velocities computed from the vibration of the pendulum.

OF THE VELOCITY OF THE BALL, AS COMPUTED FROM THE PENDULUM AND GUN.

112. The four following tables contain the mediums of the velocities of the balls, as computed for each day, for all the principal charges of powder, and for each gun separately; one table being allotted for each. In these tables, all the mediums are arranged in a continued series, in the chronological order as they occurred, and accompanied with all the circumstances necessary to be known; thus forming a fund or collection of elements, from which other arrangements and principles are to be deduced.

Each table consists of ten columns. The first column

contains the dates; the next three the state of the weather and air; namely, the 2d column the hygrometer, or state of the air as to dryness and moisture; the 3d the barometer; and the 4th the thermometer; both of which last instruments, it must be observed, were always placed in the shade, and within the house, while the experiments were made in the open air, where it was commonly much hotter than the degree shown by the thermometer. The 5th column contains the weight of the charge of powder; the 6th and 7th the weight and diameter of the ball; the 8th and 9th the velocity of the ball, the former is computed from the vibration of the pendulum, and the latter from the recoil of the gun; and finally, the 10th column contains the difference between these two velocities, which is marked with the negative sign, (−) when the velocity by the gun is the less of the two.

Daily Mediums of Experiments with the Gun n° 1.

Date	Hygro	Barom.	Ther.	Powder	Ball's wt.		Ball's diam	Veloc. by the pend	Veloc. by the gun	Diff
1783		inches	deg.	oz	oz	dr	inches	feet	feet	feet
June 30	dry	30·34	74	16	16	13	1·95	1456	1315	-141
July 17	dry	30·23	72	8	·	·	1·96	1471	1501	30
19	dry	30·12	70	2	·	·	·	797	832	35
				4	·	·	·	1109	1145	36
31	dry	30·13	69	12	·	·	·	1412	1374	-38
				16	·	·	·	1367	1334	-33
Aug 12	wet	30·00	64	16	·	12½	·	1419	1399	-20
Sept 10	dry	29·7	60	2	·	·	·	785	838	53
				4	·	·	·	1087	1122	35
				8	·	·	·	1353	1396	43
18	dry	30·08	64	8	·	13	·	1383		
				10	·	·	·	1417		
				12	·	·	·	1375		
				14	·	·	·	1333		
				16	·	·	·	1243 D		
				20	·	·	·	1144		
				24	·	·	·	1194		
				32	·	·	·	880		
				36	·	·	·	838		
30	dry	30·25	64	6	·	14	·	1331		
				8	·	·	·	1386		
				10	·	·	·	1402		
				12	·	·	·	1453		
				14	·	·	·	1402		
1784										
Aug 4	wet			6	·	·	·	1295	1339	44
				6	·	·	·	1368		
11	hazy	30·25	65	8	·	15	1·97	1475		
				10	·	15	·	1493		
				12	·	14⅔	·	1520		
				14	·	14⅔	·	1528		
Sept 10	fair			2	·	12	1·96	755		
				4	·	·	·	1131	1170	39
				6	·	·	·	1370	1358	-12
				8	·	·	·	1475		
21	fair			4	·	12	1·97	1124		
				6	·	12	1·97	1372		
				8	·	11	1·96	1445		
Oct. 4	dry			2	·	13	1·96	759		
				4	·	12	1·96	1086		
				6	·	12	1·96	1325		
				8	·	12	1·96	1472		
5	dry			8	·	13	1·96	1411		
6	dry			8	·	9	1·95	1436		
11	hazy			8	·	7	1·95	1444		

Daily Mediums of Experiments with the Gun n° 2.

Date	Hygrom.	Barom.	Ther.	Powder	Ball's wt		Ball's diam	Velocity by the pend	Velocity by the gun	Diff
1783		inches	deg	oz	oz	dr	inches	feet	feet	feet
July 23	dry	29·88	70	2	16	13½	1·96	793	840	47
				4	·	13½	·	1135 D	1207	72
				8	·	13	·	1566	1592	26
				16	·	13	·	1660	1499	-161
Aug 12	wet	30·00	64	16	·	12½	·	1676	1497	-179
Sept 11	dry	29·93	60	2	·	·	·	856	846	-10
				4	·	·	·	1239	1220	-19
				8	·	·	·	1571	1452	-119
25	dry	29·93	59	8	·	12	·	1569		
				10	·	·	·	1608		
				12	·	·	·	1615		
				14	·	·	·	1517 D		
				16	·	·	·	1664 D		
29	dry	30·28	64	6	·	11½	·	1448		
				8	·	·	·	1561		
				10	·	·	·	1618		
				12	·	·	·	1669		
				14	·	·	·	1662		
				16	·	·	·	1637		
				18	·	11	·	1598		
				20	·	·	·	1639 D		
1785										
Sept 2	cloudy			8	·	13	1·96	1503		
9	dry			4	·	12	1·96	1204		

Daily Mediums of Experiments with the Gun n° 3.

Date	Hygrom.	Barom	Ther.	Powder	Ball's wt		Ball's diam	Velocity by the pend	Velocity by the gun	Diff
1783		inches	deg	oz	oz	dr	inches	feet	feet	feet
July 12	dry			16	16	13	1·96	2030	1706	-324
18	dry	30·28	68	4	·	·	1 96	1353	1321	-32
				8	·	·	·	1766	1620	-146
19	dry	30·12	70	2	·	·	·	898	921	23
Aug 13	cloudy	30·17	64	8	·	12½	·	1803	1594	-209
				16	·	·	·	1966	1542	-424
Sept 8	moist	30·03	61	2	·	13	·	926	928	2
				4	·	·	·	1334	1266	-68
1784										
Aug 5	dry	29·98	68	6	·	14	1·97	1616		
Sept 11	dry			4	15	2	1·87	1225		
				4	16	2	1·92	1244		
				4	16	14	1·97	1346		
				8	15	2	1·87	1662		
				8	16	3	1·92	1728		
				8	16	14	1·97	1815		
16				4	16	9	1·96	1388 D		

Daily Mediums of Experiments with the Gun n° 4.

Date	Hygrom.	Barom.	Ther.	Powder	Ball's wt		Ball's diam	Velocity by the pend	Velocity by the gun	Diff
1783		inches	deg	oz	oz	dr	inches	feet	feet	feet
July 29	dry	29·90	72	8	16	13	1·96	1936	1643	-293
				16	·	·	·	2161	1656	-505
30	dry	30·06	69	2	·	·	·	968	929	-39
				4	·	12	·	1375	1295	-80
1784										
Oct 12	dry			16	·	11	·	2060		

113. The foregoing tables contain the several mediums of velocities, for each day, and for all varieties in the circumstances of powder, and weight and diameter of ball. It will now then be proper to collect together all the repetitions of the same charge or weight of powder, and to take the mediums of all those mediums, to serve as fixed radical numbers, or established degrees of velocity, adapted to all the various charges of powder, and length of gun. Now, for this purpose, it may be proper to reduce the numbers of these tables all to one common weight and diameter of ball, namely, to the weight 16 oz 13 dr, and the diameter 1·96 inches, which are the numbers that most commonly occur. And this reduction will be very well deduced from the experiments of September 11, 1784, when several trials were made with divers weights and diameters of ball, and with both 4 oz and 8 oz of powder, the results of which accord very well together. In the experiments of that day, it was found that, with the 4 oz charges, $\frac{1}{7}$ of the whole velocity is lost by the difference of $\frac{1}{10}$ of an inch in the diameter of the ball; and, with the 8 oz charge, $\frac{4}{15}$ of the velocity is lost by the same difference of windage. But the quantity of inflamed fluid which escapes, will be nearly as the difference between the area of the circle of the bore and the great circle of the ball, or the force will be as the square of the ball's diameter; and the velocity, we know, is as the square root of the force: therefore the velocity is as the diameter of the ball; and the difference in the velocity, as the difference in the diameter, or as the windage. Hence, if w denote any difference of windage in parts of an inch, or difference between 1·96 and the diameter of any ball, and $\frac{1}{m}$ the part of the experimented velocity lost by $\frac{1}{10}$ of an inch difference of windage; then shall $\frac{1}{10} : w :: \frac{1}{m} : \frac{10w}{m}$, which last term will show what part of the experimented velocity is lost by the increase of windage denoted by 10. By this rule then, I reduce all the velocities to what they would have been, had the diameter of the ball been con-

stantly 1·96. It is to be noted, however, that the value of m will vary with the charge of powder: with 4 ounces of powder, it was found that $\frac{1}{m}$ was $\frac{1}{7}$ of the whole velocity, or $\frac{1}{6}$ of the experimented velocity; but with 8oz of powder, $\frac{1}{m}$ was found to be $\frac{2}{15}$ of the whole, or $\frac{2}{13}$ of the experimented velocity. We shall not be far from the truth, therefore, if we take the following values of $\frac{1}{m}$, to the several corresponding charges of powder; that is, as far as 16 oz in the guns n° 3 and 4, and then returning backwards again, as the powder is increased above 16 oz, by 2 oz at a time; but in the gun n° 2, to continue only to 14 oz, and then return backwards again for all above 14 oz; and for the gun n° 1, to continue only to 12 oz, and then return backwards for all above that charge.

Powder					Value of $\frac{1}{m}$
2	$\frac{4}{23}$	or	$\frac{2}{11}$	=	·182
4	$\frac{4}{24}$	=	$\frac{1}{6}$	=	·166
6	$\frac{4}{25}$	=	$\frac{4}{25}$	=	·160
8	$\frac{4}{26}$	=	$\frac{2}{13}$	=	·154
10	$\frac{4}{27}$	=	$\frac{4}{27}$	=	·148
12	$\frac{4}{28}$	=	$\frac{1}{7}$	=	·143
14	$\frac{4}{29}$	=	$\frac{4}{29}$	=	·138
16	$\frac{4}{30}$	=	$\frac{2}{15}$	=	·133

Such then is the reduction of the velocity on account of the windage. And as to that for the different weights of the ball, we know that the velocity varies in the reciprocal subduplicate ratio of the weight; and according to this rule the numbers were corrected on account of the different weights of ball. After these reductions then are made, the numbers in the foregoing tables, arranged under their respective charges of powder, will be as follow, for a ball of 1·96 diameter, and weighing 16 oz 13 dr.

Mean Velocities of Balls, for all the Guns, with several Charges of Powder, reduced to a Ball of 1·96 *Diameter, and weighing* 16 oz 13 dr.

Powder, oz	2	4	6	8	10	12	14	16
	797	1109	1334	1471	1417	1412	1333	1478
	784	1086	1298	1352	1405	1375	1405	1367
	754	1129	1371	1383	1476	1453	1511	1418
	759	1103	1368	1389		1503		1243D
		1084	1347	1458				
GUN n° 1			1322	1472				
				1439				
				1469				
				1411				
				1447				
				1449				
mediums	774	1102	1340	1431	1433	1436	1416	1430
	794D	1136D	1444	1566	1605	1612	1657	1660
	855	1238		1569	1613	1664		1674
GUN n° 2		1204		1566				1661
				1557				1632
				1503				
mediums	825	1191	1444	1552	1609	1638	1657	1656
	898	1353	1593	1766				2030
	926	1334		1801				1966
GUN n° 3		1327		1793				
		1378D						
mediums	912	1348D	1593	1787				1998
	968	1373		1936				2161
GUN n° 4								2052
mediums	968	1373		1936				2200

114. These last medium velocities, for each gun, will be tolerably near the truth; and the more so, commonly, as the number of the other mediums is the greater. For want, however, of a sufficient number of each sort, there are some small irregularities among the final mediums, which may be corrected, for the most part, by adding or subtracting 3 or 4 feet, as they are sometimes too little, and sometimes too great. And these small deviations will be very easily disco-

vered by dividing the mediums by each other, namely, each of the velocities for 4, 6, 8, &c, ounces of powder, by that for 2 ounces. For we know, from the principles of forces, and other experiments, that the velocities will be nearly as the square roots of the quantities of powder; that is, while the length of the charge does not much shorten the length of the bore before the ball; but gradually deviating from that proportion more and more, as the charge of powder is increased in length; because the force has gradually a less distance and time to act upon the ball in. Now by dividing the quantities of powder 4, 6, 8, &c, by 2, the quotients 2, 3, 4, &c, show the ratios of the charges; and the roots of these numbers, namely,

1·414
1·732
2·000
&c,

show the ratios which the velocities would have to each other nearly, if the empty part of the bore was constantly of the same length. But as the vacant part always decreases as the charge increases, the ratios of the velocities may be expected to fall short of those above, and the sooner and the more so, as the gun is shorter. Accordingly, on trial, we find the ratios hold pretty well, even in the shortest gun, as far as to the 6 oz charge; but in the 8 oz charge it falls about $\frac{1}{13}$ or $\frac{1}{14}$ part below the true ratio, being 1·85 instead of 2. In the longer guns, the proportions hold out gradually longer, and the deviations are always less and less; thus, in the 2d gun, the ratio for the 8 oz charge is about 1·895, in the 3d it is 1·945, and in the 4th gun it is 1·999 or 2 very nearly. And so for other charges. Correcting then some of the mediums by means of this property, the more accurate radical medium velocities, for each gun, with the several charges of 2, 4, 6, and 8 ounces of powder, will be as here follow:

Powder	Gun n° 1					N° 2				
	Ratio.	Veloc.	Dif. 1.		Dif. 11.	Ratio.	Veloc.	Dif. 1.		Dif. 11.
2		780		320			835		345	
4	1·410	1100		240	80	1·414	1180		265	80
6	1·731	1340		90	150	1·730	1445		185	130
8	1·850	1430				1·893	1580			
Powder	Gun n° 3					N° 4				
2		920		380			970		400	
4	1·413	1300		290	90	1·412	1370		310	90
6	1·729	1590		200	90	1·732	1680		260	50
8	1·945	1790				2·000	1940			

where the velocity is set in large characters in the middle column; and on the left hand, in a small character, is the ratio, which is found by dividing each velocity by the first, the law of which ratios has been mentioned above; also on the right hand are the columns of first and second differences; the first being the difference between each two succeeding numbers, and the second the differences of those differences.

Or, for some purposes, it may be more convenient to range the velocities, &c, as here below:

Gun n°	2 oz				4 oz			
1			780	55	1·410	1100		80
2			835	85	1·414	1180		120
3			920	50	1·413	1300		70
4			970		1·412	1370		
Gun n°	6 oz				8 oz			
1	1·731	1340		105	1·850	1430		150
2	1·730	1445		145	1·893	1580		210
3	1·729	1590		90	1·945	1790		150
4	1·732	1680			2·000	1940		

where the numbers are here placed in horizontal lines, which before were vertical; and vertical here, those which before were horizontal: and where the law, both of the ratios and differences, is evident. We also hence perceive how, for each charge, the velocity of the ball is continually increased as the gun is longer.

And these velocities may be considered as standard radical numbers, here deposited, and ready to be applied to any purpose, in which the consideration of the velocity can be useful. And those for the other charges of powder will be as in the general table in Art. 113.

115. These velocities, however, it must be remarked, are those with which the ball strikes the pendulum, after passing through the air between it and the muzzle of the gun; and consequently they are less than the velocities with which it immediately issues from the gun, by as much velocity as the ball loses by the resistance of the air, in its flight through that space. Now we have found, in Art. 33, that the first velocities lose at least their 84th part by that resistance, when the air behind the ball is supposed instantly to fill up the place always quitted by the ball in its flight. But as this is not exactly the case, the air rushing into a vacuum with a certain finite velocity only, therefore the part lost will be gradually more and more as the ball moves swifter, till its velocity become equal to that of the air itself; after which the part lost will remain constant. And Mr. Robins asserts, that the velocity lost by very swift motions, is about 3 times as great as that lost by slow ones; and therefore that will be about the 28th part. So that the loss will always lie between the 84th part and the 28th part. I shall therefore leave it in this uncertain state, till other experiments enable us to ascertain what may be the exact proportion of loss peculiar to every degree of velocity.

116. From the general table of medium velocities in Art. 113, it is evident that, for each gun, the velocity increases with the charge to a certain extent, where it is greatest; and that afterwards it gradually decreases as the charge is increased. It further appears that the point, or charge, at which the velocity is the greatest, is different in the guns of different lengths; the charge which gives the maximum of velocity, being always greater, as the gun is longer. And by tracing this increase of charge, from the beginning, to the point of greatest velocity, it appears that,

with the 1st, 2d, and 3d guns, the charges which give the greatest velocities, are nearly as follow, viz.

Gun n° 1 at the charge of 12 oz,
- - 2 - - - - - 14 oz,
- - 3 - - - - - 16 oz.

Here it will not be so proper to specify what portion of the weight of the ball these weights of powder are; being no ways regulated by that circumstance; but what portion of the bore of the gun is filled with these quantities of powder. Now, by the table of the lengths of charges in Art. 104, it appears that the lengths of the charges of 12, 14, and 16 oz, are these, viz.

12 oz - -	8·20 inches;	gun 1, its length	28·5,
14 oz - -	9·47 inches;	gun 2 - - -	38·4,
16 oz - -	10·74 inches;	gun 3 - - -	57·7.

Then dividing each length of charge by its corresponding length of gun, we obtain nearly these three following fractions, viz.

$\frac{3}{10}$ in gun 1 of 15 calibers long,
$\frac{1}{4}$ or $\frac{3}{12}$ in gun 2 of 20 calibers long,
$\frac{3}{16}$ in gun 3 of 30 calibers long,

which express what part of the bore is filled with powder, when the greatest velocity is given to the ball, with each of these lengths of gun. And which therefore is not one and the same constant part for all lengths of gun, but varying nearly in the reciprocal subduplicate ratio of the length of the bore; or still nearer in the reciprocal subduplicate ratio of the empty part of the bore before the charge. And, by this rule, finding the part for the longest gun, or n° 4, it will be found to be nearly $\frac{3}{20}$ or 12 inches in length, answering to 18 ounces of powder. So that the whole set of numbers, for the greatest velocity, will be as follows:

Gun n°	Length of bore	The Charge		
		Wt. oz	Length	
			Inches	Part of whole
1	28·5	12	8·2	$\frac{2}{7}$
2	38·4	14	9·5	$\frac{1}{4}$
3	57·7	16	10·7	$\frac{2}{11}$
4	80·2	18	12·0	$\frac{1}{20}$

117. Having so far settled the degree of velocity of the ball, as determined by the vibration of the pendulum, we may in like manner now proceed to assign the mean velocities, as deduced from the recoil of the gun. The repetitions in this latter way are not so numerous as in the former; but, such as they are, we shall here abstract them from the general tables in Art. 112, reducing them, however, all to the same common weight and diameter of ball, as was done in Art. 113.

Mean Velocities from the Recoil of the Gun.

Powder, oz	2	4	6	8	12	16
Gun n° 1	832	1145	1344	1501	1374	1337
	837	1120	1352	1393		1394
		1165				1396
mediums	835	1143	1348	1447	1374	1356
Gun n° 2	841	1209		1592		1499
	845	1218		1450		1494
mediums	843	1213		1521		1496
Gun n° 3	921	1321		1620		1706
	928	1266		1591		1540
mediums	925	1294		1605		1623
Gun n° 4	929	1293		1643		1656

These mediums however are not so exact as those in Art. 111, because those were deduced from a greater number of particulars. We shall therefore chiefly adopt those that were stated in that article, for the radical standard velocities of the ball, as determined from the recoil of the gun, excepting in some instances when the other is used, and sometimes the mediums of both. So that the final mediums will be as follow:

Velocities of the Ball from the Recoil of the Gun.

Gun n°	2 oz	4 oz	8 oz	16 oz
1	830	1135	1445	1345
2	863	1203	1521	1485
3	919	1294	1631	1680
4	929	1317	1669	1730

118. Let us now compare these velocities, deduced from the recoil of the gun, with those that are stated in Art. 113 and 114, which were determined from the pendulum; that we may see how near they will agree together. And, in this comparison, it will be sufficient to employ the velocities for 2, 4, 8, and 16 ounces of powder; this will be the most certain also, as these mediums are better determined than most of the others.

Comparison of the Velocities by the Gun and Pendulum.

Gun n°	2 oz			4 oz		
	Velocity by		Diff.	Velocity by		Diff.
	gun	pend		gun	pend	
1	830	780	50	1135	1100	35
2	863	835	28	1203	1180	23
3	919	920	-1	1294	1300	-6
4	929	970	-41	1317	1370	-53
Gun n°	8 oz			16 oz		
1	1445	1430	15	1345	1377	-32
2	1521	1580	-59	1485	1656	-171
3	1631	1790	-159	1680	1998	-318
4	1669	1940	-271	1730	2106	-376

In this table, the first column shows the number of the gun; and its velocity of ball, both by the vibration of the gun and pendulum, with their differences, is on the same line with it, for the several charges of powder. After the first column, the rest of the page is divided into four spaces, for the four charges, 2, 4, 8, 16 ounces; and each of these

is divided into three columns: in the first of the three, is the velocity of the ball as determined from the vibration of the gun; in the second is the velocity as determined from the vibration of the pendulum; and in the third is the difference between the two, which is marked with the negative sign, or $-$, when the former velocity is less than the latter, otherwise it is positive.

119. From the comparison contained in the last article, it appears, in general, that the velocities, determined by the two different ways, do not agree together; and that therefore the method of determining the velocity of the ball from the recoil of the gun, is not generally true, though Mr. Robins and Mr. Thompson (now Count Rumford) had suspected it to be so; and consequently that the effect of the inflamed powder on the recoil of the gun, is not exactly the same when it is fired without a ball, as when it is fired with one. It also appears that this difference is no ways regular, neither in the different guns with the same charge, nor in the same gun with different charges of powder. That with very small charges, the velocity by the gun is greater than that by the pendulum; but that the latter always gains upon the former, and soon becomes equal to it; after which, it exceeds it more and more as the charge of powder is increased. That the particular charge, at which the two velocities become equal, is different in the different guns; and that this charge is less, or the equality sooner takes place, as the gun is longer. And all this, whether we use the actual velocity with which the ball strikes the pendulum, or the same increased by the velocity lost by the resistance of the air, in its flight from the gun to the pendulum.

OF THE RANGES AND TIMES OF FLIGHT.

120. Having dispatched what relates to the velocity of the ball, we may now proceed in like manner to the expe-

riments made to determine the actual ranges, and the times of flight of the balls.

The mediums of these, hitherto obtained, are not so numerous as could be wished; however, such as they are, we shall here collect them, in the same manner as we did the circumstances relating to the initial velocities in Art. 112.

Mediums of Ranges and Times of Flight.

Date	Barom	Ther.	Powder	Ball's wt		Ball's diam	Elevat gun	Time flt	Range
1785	inches	deg	oz	oz	dr	inches	deg	sec	feet
Sept 2	29·80	66	8	16	13	1·965	15	14·0	5916
8	30·02	65	8	16	12	1·96	15	14·7	6216
14	30·50	67	4	16	12	1·96	15	8·4	4398
23	30·35	60	4	16	10	1·963	15	8·3	4523
			2	16	8	1·97	45	22·0	5068
29	30·35	60	2	16	12	1·95	45	20·3	5150
			12	16	12	1·95	15	15·5	6700
1786									
June 12	29·89	63	4	16	3	1·957	15	11·0	5060
			2	16	5	1·959	15	9·2	4130
1785									
Oct 4	29·93		8	15	3	1·96	15		5600
11	29·88	60	8	15	12	1·96	15	10·1	5620

Of these, the first 6 day's experiments were with the gun n° 2; and the last two days, with the gun n° 3.

121. Now, by taking again the mediums of these, both in the balls, and their ranges and times of flight, they will finally come out as follows:

Final Mediums of Ranges and Times.

Gun	Powder	Ball's wt		Ball's diam	Elevat gun	Time flt	Range	Velocity of ball
	oz	oz	dr	inches	deg	sec	feet	feet
N° 2	2	16	10	1·96	45	21·2	5109	863
	2	16	5	1·959	15	9·2	4130	868
	4	16	8⅓	1·96	15	9·2	4660	1234
	8	16	12½	1·962	15	14·4	6066	1644
	12	16	12	1·95	15	15·5	6700	1676
N° 3	8	15	7½	1·96	15	10·1	5610	1930

And in the last column are added the corresponding initial velocities, which the ball would have at the muzzle of the gun; which have been extracted from the medium velocities, as determined by the pendulum, and here reduced to the peculiar weight and diameter of ball in each particular case of this table, by the reductions specified in Art. 113, and by augmenting the velocity for the 2 ounce charge by its 36th part, and the others by their 28th part, for the loss of velocity in passing from the gun to the pendulum.

So that, in this little table, we have the following concomitant data, determined with a tolerable degree of precision; namely, the weight of powder, the weight and diameter of the ball, the initial or projectile velocity, the elevation of the gun, the time of the ball's flight, and its range, or the distance on the horizontal plane. From which it is hoped that the resistance of the medium, and its effect on other elevations, &c, may be in some measure determined, and so afford the means of deriving rules for the several cases of practical gunnery: a subject intended to be further treated on, in the future prosecution of these experiments.

OF THE BALL'S PENETRATION INTO THE WOOD.

I shall here select only those depths of the penetrations into the block of wood, which have been made in the course of

the last year's experiments, as they are the most numerous and uniform, and were all made with the same gun, namely, n° 2. I shall also select only those for 2, 4, and 8 ounces of powder, as they are the most useful and certain numbers, for affording safe and general conclusions; and besides, the trials with other charges are too few in number, being commonly no more than one of each.

Mean Penetrations of Balls into Elm Wood.

Powder 2	4	8
7	16·6	18·9
	13·5	21·2
		18·1
		20·8
		20·5
means 7	15	20

That is, the balls penetrated about

7 inches deep with 2 oz of powder
15 - - - - - - 4 - - - - -
20 - - - - - - 8 - - - - -

And these penetrations are nearly as the numbers 2, 4, 6, or 1, 2, 3; but the quantities of powder are as 2, 4, 8, or 1, 2, 4; so that the penetrations are as the charges as far as 4 ounces, but in a less ratio at 8 ounces, namely, less in the ratio of 3 to 4. And are indeed, so far, proportional to the logarithms of the charges. But the wood was wet within side, and therefore probably soft, which will give the penetrations too much.

Now, by the theory of penetrations, the depths ought to be as the charges, or, which is the same thing, as the squares of the velocities. But from our experiments it appears, that the penetrations fall short of that proportion in the higher charges. And therefore it would seem, that the resisting force of the wood is not uniformly the same; but that it increases a little with the increased velocity of the ball.

And this probably may be occasioned by the greater quantity of fibres driven before the ball; which may thus increase the spring or resistance of the wood, and so prevent the ball from penetrating so deep as it otherwise would do. But it will require further experiments in future to determine this point more accurately.

122. Before concluding this account, it may not be unuseful to make a short recapitulation of the more remarkable deductions that have been drawn from the experiments, in the course of these calculations. For, by bringing them together into one collected point of view, we may, at any time, easily see what useful points of knowledge are hereby obtained, and thence be able to judge what remains yet to be done by future experiments. Having therefore experimented, and examined, all the objects that were pointed out in Art. 5, we shall just slightly mention the answers to these enquiries; which are either additions to, or confirmations of, those laid down in Art. 2, as drawn from the former experiments in the year 1775.

And 1st, then, it may be remarked, that the former law, between the charge and velocity of ball, is again confirmed, namely, that the velocity is directly as the square root of the weight of powder, as far as to about the charge of 8 ounces: and so it would continue for all charges, were the guns of an indefinite length. But as the length of the charge is increased, and bears a more considerable proportion to the length of the bore, the velocity falls the more short of that proportion.

2nd. That the velocity of the ball increases with the charge, to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but not greater however, in so high a proportion as the length of the gun is; so that the part of the bore filled with powder bears a less proportion to the whole in the long guns, than it does

in the shorter ones; the part of the whole which is filled being indeed nearly in the reciprocal subduplicate ratio of the length of the empty part. And the other circumstances are as in this tablet.

Table of Charges producing the Greatest Velocity.

Gun n°	Length of the bore	Length filled	Part of the whole	Weight of the powder
	inches	inches		oz
1	28·5	8·2	$\frac{4}{14}$	12
2	38·4	9·5	$\frac{4}{16}$	14
3	57·7	10·7	$\frac{4}{22}$	16
4	80·2	12·0	$\frac{4}{27}$	18

3dly. It appears that the velocity continually increases as the gun is longer, though the increase in velocity is but very small in respect to the increase in length, the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but somewhat greater than that of the cube roots of the length, and is indeed nearly in the middle ratio between the two. But the particular degrees of velocity for each gun, and charge, may be seen at p. 67 and 69.

4thly. It appears from the table of ranges in Art. 121, p. 76, that the range increases in a much less ratio than the velocity, and indeed is nearly as the square root of the velocity, the gun and elevation being the same. And when this is compared with the property of the velocity and length of gun in the foregoing paragraph, it appears that we gain extremely little in the range by a great increase in the length of the gun, the charge being the same. And indeed the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about $\frac{1}{7}$th part more range for a double length of gun.

5thly. From the same table in Art. 121, it also appears, that the time of flight is nearly as the range; the gun and elevation being the same.

6thly. It appears that there is no difference caused in

the velocity or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it.

7thly. But a very great difference in the velocity arises from a small degree of windage. Indeed with the usual established windage only, namely, about $\frac{1}{20}$th of the caliber, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost. And as the balls are often smaller than that size, it frequently happens that $\frac{1}{2}$ the powder is lost by unnecessary windage.

8thly. It appears that the resisting force of wood, to balls fired into it, is not constant. And that the depths penetrated by different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity.

9thly. These, and most other experiments, show, that balls are greatly deflected from the direction they are projected in; and that so much as 300 or 400 yards in a range of a mile, or almost $\frac{1}{4}$th of the range, which is nearly a deflection of an angle of 15 degrees.

10thly. Finally, these experiments furnish us with the following concomitant data, to a tolerable degree of accuracy; namely, the dimensions and elevation of the gun, the weight and dimensions of the powder and shot, with the range and time of flight, and first velocity of the ball; from which it is to be hoped, that the measure of the resistance of the air to projectiles may be determined, and thereby the foundation be laid for a true and practical system of gunnery, which may be as well useful in service as in theory; especially after a few more accurate ranges are determined, with better and larger balls than some of the last employed on the foregoing ranges.

THE EXPERIMENTS OF 1787.

125. The chief object of this year's experiments was, by discharging the balls against the ballistic pendulum, placed at different distances, to observe how much velocity would be lost in passing through different spaces of air, and thence to determine the quantity of the air's resistance, to every degree of velocity in the ball. The gun employed was n° 2: and the charges of powder were 2 ounces and 8 ounces; because the velocity of the ball with the former charge would be about 800 or 900 feet, and of the latter about 1600 or 1800 feet per second; nearly in the middle between which, Mr. Robins says, the law of the resistance changes from the square of the velocity, so as to give about triple the result of that ratio.

Three barrels of fresh good powder were mixed together, to be used in these experiments, that they might be all made with the same kind, or powder of equal strength. And a parcel of very nice round balls were cast, with very small windage, being each of 2 inches in diameter, and weighing 18 ounces, sometimes a dram or two over or under. The charges, as before, were put in flannel bags, and moderately thrust up always to the same height in the bore.

126. *Monday, September* 10, 1787.

Attended to weigh the pendulum, and measure the distances to the centres of gravity and oscillation; which were as below:

The weight of the pendulum and its spear 791 lb.
Distance to the centre of gravity - - - 78·2 inches
Vibrations in 10 min. by a medium of 2 times 401.

Hung up the pendulum in its place, and prepared every thing, to be ready to begin firing the next morning.

127. *September* 11, 1787; *from* 10 *till* 1.

Dry weather.

Barometer 30·33; Thermometer 58°.

N°	Powder	Balls		Dist. gun	Vibr pend	Point struck	Plugs wt	Values of		Velocity of ball
								p	g	
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	8	18	0	30	180	88·0	3	791·0	78·2	1728
2	2	18	0	30	79	89·0	3			
3	8	18	0	30	173	89·4	3	793·4	78·2	1638
4	2	18	0	30	90	90·2	3	794·5	78·3	847 D
5	2	18	0	30	103	89·2	3	795·7	78·3	981
6	2	18	0	30	99	87·7	4	796·9	78·3	961
7	8	18	0	30	177	86·8	4	798·1	78·3	1738
8	8	18	0	90	157	87·9	4	799·3	78·4	1527 D
9	2	18	0	90	95	88·7	4	800·4	78·4	917
10	8	18	0	90	165	91·9	3	801·6	78·4	1539 D
11	2	18	0	90	96	86·6	3	802·8	78·4	952
12	8	18	0	90	165	86·9	5	804·0	78·4	1632
13	2	18	0	90	93	84·9	4	805·2	78·5	934

To find the mediums.

	With 8 oz		2 oz	
	At 30 feet	90 feet	30 feet	90 feet
	1728	D	D	917
	1638	D	981	952
	1738	1632	961	934
Medi.	1702	1632	971	934
Lost in 60 feet	-	70	- - - - - - -	37

Therefore, in passing through 60 feet of air, the greater lost 70, or 24th part, the less lost 37, or 26th part.

The charge at n° 2 was not set up home in the bore.

Some of the rounds are omitted as doubtful, marked D.—The number of oscillations will remain invariable, as the balls lodged about the centre of oscillation, or nearly at equal distances above and below it.——The mean of the penetrations was 8·8 inches, with 2 oz of powder.

128. *September* 12, 1787; *from* 10 *till* 1.

Dry weather.

Barometer 30·31; Thermometer 57°.

N°	Powder	Ball		Dist. gun	Vibr pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of ball
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	8	18	0	150	153	89·2	4	806·3	78·5	1481D
2	2	18	1	150	93	88·1	2	807·5	78·5	909
3	8	18	1	150	153	82·3	4	808·7	78·5	1603
4	2	18	1	150	91	89·1	3	809·9	78·6	883
5	8	18	1	150	152	87·5	4	811·1	78·6	1503
6	2	18	1	150	94	88·6	3	812·2	78·6	920
7	8	18	1	210						
8	8	18	1	210	147	84·0	5	813·4	78·6	1519
9	2	18	1	210	90	90·4	3	814·6	78·6	866
10	8	18	1	210	142	86·0	4	815·8	78·7	1439
11	2	18	1	210	75					
12	2	18	2	210	89	87·3	4	817·0	78·7	887
13	8	18	0	210	146	84·4	5	818·1	78·7	1519
14	2	18	1	210	86	83·5	3	819·3	78·7	902

	With 8 oz		2 oz	
	At 150 feet	210 feet	150 feet	210 feet
	D	1519	909	866
	1603	1439	883	887
	1503	1519	920	902
Mediums	1553	1492	904	876
Lost in 60 feet		61		28

So that, in passing through 60 feet of air, the greater lost 61, or 25th part, the less lost 28, or 32d part.

N° 7 struck the edge of one of the side iron bands, which turned the ball aside from the pendulum. And n° 11 struck the iron band, which broke the ball in pieces.—The penetration of n° 8 was 19½ inches.

129. *September* 13, 1787; *from* 10 *till* 3.

Dry weather.

Barometer 30·25; Thermometer 56°.

N°	Powder	Ball		Dist. gun	Vibr pend	Point struck	Plugs wt	Values of		Velocity of ball
								p	g	
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	8	17	15	300						
2	8	18	1	300	121	78·5		820·5	78·8	1354
3	8	18	1	300	152	92·6	8	821·7	78·8	1445
4	8	18	1	300	135	93·4	2	822·9	78·8	1273 D
5	8	18	2	300						
6	8	18	2	300	121	79·9	3	824·0	78·8	1331 D
7	8	18	2	300	122	82·0	3	825·2	78·8	1310 D
8	8	18	1	300	135	89·3	2	826·4	78·9	1339 D
9	2	17	15	300	77	84·6	2	827·6	78·9	814 D
10	2	17	15	300	93	99·1		828·8	78·9	840
11	2	17	15	300	83	90·4	2	829·4	78·9	823
12	2	17	15	300	89	95·3	2	830·5	78·9	838
13	8	17	15	378	134	94·4	3	831·7	79·0	1277
14	2	17	15	378	85	95·0	3	832·9	79·0	806
15	8	17	15	378				834·0	79·0	

	With 8 oz		2 oz	
	At 300 feet	378 feet	300 feet	378 feet
Mediums	1400	1277	834	806
Lost in 78 feet	-	123	- - - - - - - -	28

So that, in passing through 78 feet of air, the greater lost 123 feet, or 11th part, the less lost 28 feet, or 30th part. But the velocity 1277, in the 13th round, is probably too small.

The 5th and 15th rounds missed the block. The 10th ball struck the bottom bar, and broke it: the ball also broke in two; the one half of which lodged in the block, and the other half rebounded 3 or 4 feet back only.

After the experiments were finished this day, the pendulum was measured and weighed, as at first; when it was found that the

Distance to the centre of gravity was at 79·0 inches or - - - - - - - -	$\frac{3}{10}$	more than at first;
N° of oscillations in 10 minutes by two watches, was - - - - - - - -	401	the same as at first;
Weight of the pendulum and spear - -	834 lb,	
Now the balls lodged were 36¼ at 18oz each	41	
And the weight of plugs was - - - -	7½	
Therf. the wt. of the balls and plugs was	48½	
Which added to the weight at first, viz.	791	
makes	839½	

So that there has been lost by evaporation 5½ lb in 3 days, the weather being very dry.

130. *September* 20, 1787. *Fine warm day.*

Barometer 29·64; Thermometer 61°.

Experiments made in the Barrack Field, with balls of equal diameter, but of different weights and materials, discharged out of a small Coehorn mortar, to show the Resistance of the Air.

N°	Powder	Ball's wt		Ball's diam	Time of flight	Deviation	Ranges	Mediums do.	Nature of the shot
	oz	lb	oz	inches	sec	feet	feet	feet	
1	1	14	4	4·43			259		
2	1	·	4	·	4		255	256	Shells filled with lead
3	1	·	4	·	4½		254		
4	2	·	4	·	7	30 R	870	870	
5	1	12	14	4·49	5		376		
6	1	·	14	·	5		399	376	
7	1	·	14	·	5		354		Solid iron balls
8	2	·	14	·	8	L 41	1068	1068	
9	1	8	12	4·43	5		342		
10	1	·	11	·	5		345	365	
11	1	·	10	·	5		409		Shells filled with water
12	2	·	9	·	9	48 R	1083	1083	
13	1	8	5½	4·44	5½		410		
14	1	·	5½	·	6		432	426	Empty shells
15	1	·	5½	·	5½		435		
16	1	1	6	4·40	5½	L 35	487		
17	1	·	6	·	5½	61 R	460	505	Oaken balls
18	1	·	6	·	7	8 R	567		
19	2	·	6	·			not	found	
20	1	0	7¼	·	5½	71 R	311		Paper balls rubbed over with whiting, to mark the fall
21	1	·	7¼	·	5	5 R	364	325	
22	1	·	7¼	·	5	L 80	300		
23	1	·	6¼	·	5	L 58	366		
24	1	·	6¼	·	6	L 105	396	403	Ditto without whiting
25	1	·	6¼	·		130 R	447		
26	2	·	7¼	·			450	450	Do. with whitn.

The diameter of the mortar bore 4·6 inches: its elevation was always 45°.

About an ounce of water came through the wooden plug during the flight of the water shells.—The side deviations, to the right (R) and left (L) of the line of direction, with 1 oz of powder, were little or nothing in the heavy shot; but those of the light ones were very great.—It seems the resistance was much greater on the paper ball when its surface was made rough and uneven with the whiting.

In the heavier shot, it appears that the range is about tripled with the double charge of powder.

131. *Abstract of the preceding Experiments of* 1787.

By collecting now together the medium results of the three days experiments, of September 11, 12, 13, of this year 1787, are obtained the following medium velocities, of the balls of 2 inches diameter, and 18 oz weight, as discharged from the gun n° 2, with 2 oz and 8 oz of powder, when striking the pendulum at the distances expressed in the first column.

Experimented Velocities at the following distances.					Velocities for other distances, deduced from the former.				
Dists.	Velocities with				Dists.	Velocities with			
	8 oz		2 oz			8 oz		2 oz	
feet	feet		feet		feet	feet		feet	
0	.		.		0	1740		990	
30	1702		971		60	1667	73	953	37
90	1632	70	934	37	120	1597	70	919	34
150	1553	79	904	30	180	1530	67	888	31
210	1492	61	876	28	240	1466	64	859	29
300	1400	92	834	42	300	1404	62	832	27

The first of these two tables shows the collection of experimented mean velocities, with which the balls struck the pendulum, at the several irregular distances set in the first column; the 2d column shows the discharges with 8 oz of powder, and the 3d with 2 oz. The second table may also be considered as experimented velocities, with similar charges, for the several regular distances in arithmetical progression, having the common difference of 60 feet; being derived from the former by interpolations. Also, the

annexed differences of the velocities, in small figures, show the velocity lost by the balls, in passing through 60 feet of air, or the difference of the distances in the 3d column.

Now, to determine how much the resistance of the air is, to the several velocities, by which the said decreases in the velocities are produced, we may proceed as follows: Put s = any space passed over in the time t, v the velocity in the middle of that space, and v' the velocity lost in passing over that space; also r the required force of resistance; the weight of the ball being denoted by 1. Then is $t = \frac{s}{v}$ nearly; and $1 : t'' :: 32 : 32t$ the velocity generated or destroyed by gravity in the time t; therefore $32t : v' :: 1 : r = \frac{v'}{32t} = \frac{vv'}{32s}$ the force of resistance, in comparison with gravity; viz, it is $\frac{vv'}{32s}$ times the weight of the ball. Then, taking $s = 60$, the space passed over, v = to the several means between the velocities in the second columns, and v' = the several differences between them, we have the following resistances, in ounces, to the ball of just 2 inches diameter, and 18 oz weight, with their corresponding velocities in the first column.

With 8 oz of Powder.			
Velocities	Resistances	1st diffs.	2d diffs.
feet	oz		
1703	$1165\frac{1}{2}$		
		$94\frac{1}{2}$	
1632	1071		$5\frac{1}{2}$
		89	
1563	982		5
		84	
1498	898		6
		78	
1435	820		

With 2 oz of Powder			
Velocities	Resistances	1st diffs.	2d diffs.
feet	oz		
972	328		
		$29\frac{1}{2}$	
936	$298\frac{1}{2}$		2
		$27\frac{1}{2}$	
904	271		2
		$25\frac{1}{2}$	
873	$245\frac{1}{2}$		$1\frac{1}{2}$
		24	
845	$221\frac{1}{2}$		

THE PENDULUM EXPERIMENTS OF 1788.

The object of the following experiments, is again to find the resistance of the air, by discharging the balls at different distances from the pendulum, and so showing the velocity lost at each distance; in order to connect with those resistances, to slower motions, that have been deduced from experiments made with the whirling machine, in the Academy this year; and some former years.

132. *August* 6, 1788. *Weather cold and cloudy.*
Barometer 30·17; Thermometer, 64°.

No	Powder	Ball's wt		Dist. gun	Vibrat pend.	Point struck	Plugs wt	Values of		Velocity of ball
								p	g	
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	2	16	9½	30	73	86·7	1·3	862·0	79·0	840 D
2	2	·	10½	·	79	89·2	2·4	863·2	·	882
3	2	·	10½	·	79	89·1	2·5	864·3	·	884
4	2	·	13½	·	80	91·4	2·1	865·5	·	864
5	8	·	9	·	144	92·0	3·0	866·6	·	1573
6	8	·	8½	·	146	90·9	4·1	867·8	·	1620
7	8	·	10½	·	143	91·4	3·6	869·0	·	1568
8	2	·	8½	60	67	85·4	3·3	870·1	·	793 D
9	2	·	7½	·	70	85·4	2·1	871·3	·	833
10	2	·	7	·	72	84·2	2·1	872·4	79·1	872
11	2	·	10	·	73	87·4	1·5	873·6	·	844
12	8	·	10½	·	146	90·3	3·5	874·8	·	1633
13	8	·	9½	·	138	90·4	3·0	875·9	·	1550
14	8	·	9½	·	124	85·0	3·4	877·1	·	1483
15	8	·	8½	·	133	88·8	2·6	878·2	·	1530
16	2	·	11	120	73	89·7	2·1	879·4	·	825
17	2	·	6½	·	75	89·0	3·0	880·6	·	869
18	2	·	7½	·	72	88·5	1·4	881·7	·	837
19	8	·	8½	·	145	95·8	3·4	882·9	·	1555
20	8	·	15½	·	140	90·3	4·1	884·0	·	1554
21	8	·	12½	·	130	90·9	3·5	885·2	·	1450
22	8	.	6	·	128	90·0	4·5	886·4	·	1480

Besides these 22 rounds, two others were fired, which missed the pendulum. At beginning this day, the pendu-

lum weighed 862 lb. Its centre of gravity was at 79 inches; and its medium vibrations were 40¼ in a minute. The gun was n° 2. The mean weight of balls was 16 oz 10 dr, and their diameter 1·965 inches.—The mediums of these sets of velocities, with 8 oz and 2 oz of powder, and at the several distances of 30 and 60 and 120 feet, are as follow:

With 8 oz			2 oz
1587 at	30 feet	distance	877
1550 at	60 feet	- -	850
1510 at	120 feet	- -	844

133. *August* 7, 1788; *Weather as before.*

Barometer 30·13; Thermometer 60°.

This day's experiments were a continuation of those of yesterday, with the same gun n° 2, and the same charges, of 2 and 8 oz of powder, but at different distances from the pendulum, viz, at 180, 240, 300 feet distances.

N°	Powder	Ball's wt		Dist gun.	Vibr. pend	Point struck	Plugs wt	Values of		Velocity of ball
								p	*t*	
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	2	16	7½	180	76	97 9	3·4	887·7	79·1	804
2	2	17	0	•	73	90·4	2·1	888·9	•	814
3	2	16	15	•	71	89·7	1·3	890·1	•	800
4	2	17	0	•	65	82·3	2·3	891·4	•	799
5	8	16	7	•	124	88·6	2·4	892·6	•	1458
6	8	17	0	•	135	93·3	4·2	893·8	•	1466
7	8	16	10	•	114	93·4	3·1	895·0	79·2	1264 D
8	8	16	7½	•	133	96·4	2·7	896·3	•	1445
9	2	16	11½	240	71	92·8	4·5	897·5	•	787
10	2	16	8	•	72	98·6	3·4	898·7	•	767
11	2	16	7	•	59	79·9	2·2	899·9	•	776
12	2	16	8	•	65	85·0	4·9	901·1	•	805
13	8	16	8	•	113	87·6	3·1	902·4	•	1360
14	8	16	14	•	121	90·2	3·4	903·6	•	1376
15	8	16	15	•	115	92·1	2·8	904·8	•	1283 D
16	8	16	13	•	106	84·5	3·0	906·0	•	1303 D
17	2	16	11	300	66	93·9	1·4	907·3	•	736 D
18	2	16	6	•	65	88·6	2·0	908·5	•	786
19	2	17	0	•	62	82·9	2·6	909·7	•	772

Besides the above 19 rounds, many others were fired, most of them at 300 feet distance, which are not here recorded, as they missed the pendulum.—The mediums of these sets of velocities, with the charges of 8 and 2 oz, and at the three distances 180, 240, 300, are as below.—The medium weight of the balls, 16 oz 11·26 dr.

With 8 oz		2 oz
1456	at 180 feet distance	804
1368	at 240 - - - -	784
·	at 300 - - - -	765

134. *August* 9, 1788. *Weather as before.*
Barometer 30·16; Thermometer 60°.
A further continuation of the same experiments.

N°	Powder	Ball's wt		Dist. gun	Vibr. pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of ball
	oz	oz	dr	feet	c	inches	oz	lb	inches	feet
1	2	16	10½	300	68	93·5	2·2	910·4	79·2	766
2	2	16	11	·	64	87·1	3·5	911·7	·	775
3	2	16	11	·	61	83·8	2·6	912·9	·	769
4	8	16	10	·	112	96·7	5·0	914·2	·	1225 D
5	8	16	10	·	108	86·9	2·4	915·4	·	1317
6	8	16	12	·	93	80·6	1·5	916·7	·	1224 D
7	8	16	11	·	108	92·3	2·6	917·9	79·3	1241
8	2	16	11	360	66	96·5	3·3	919·2	·	727
9	2	16	12	·	56	82·6	1·7	920·4	·	721
10	2	16	9	·	62	90·3	3·0	921·7	·	731
11	2	16	11	·	62	87·0	2·0	922·9	·	760
12	8	16	11	·	117	93·8	2·9	924·2	·	1332 D
13	8	16	10	·	95	80·0	2·7	925·4	·	1270
14	8	16	11	·	103	89·9	2·4	926·7	·	1227
15	8	16	9	·	87	82·2	3·0	927·9	·	1134 D

Besides these 15 rounds, a great many more were fired, which missed the pendulum.—The mediums of these velocities, as usual, were as below.—And the medium weight of the balls 16 oz 10⅔ dr.

With 8 oz		2 oz
1317	at 300 feet distance	770
1276	at 360 - - - -	735

135. Now collecting together the medium numbers, for the same charge and distance, of the last three day's experiments, omitting some that are evidently irregular, they come out as below: viz,

With 2 oz			8 oz	
Dist.	Veloc	Diffs	Veloc	Diffs
30	877	27	1587	37
60	850	6	1550	40
120	844	40	1510	54
180	804	20	1456	88
240	784	19	1368	51
300	765	30	1317	41
360	735		1276	

Here it is evident, from the columns of differences, that the numbers for the velocities are not far from the regular law which nature observes; indeed as nearly regular as such experiments can well be expected to produce. Hence we can easily infer, that the velocity at the muzzle of the gun, with the 2oz charge, was a little above 880, and with 8 oz about 1640. By making the comparison, we perceive that the numbers are all a little less than the corresponding numbers of last year, owing partly to the greater windage, and perhaps partly to the powder losing of its quality since the last year; for it was the remains of the same powder as was then used.

By making some small and obvious alterations in the above numbers, sometimes adding and sometimes subtracting a small quantity, the velocities are easily made regular, as in the following table, which may be considered as exhibiting the correct velocities with which the pendulum was struck at the several annexed distances. The adjacent numbers are also subtracted from one another, to show the velocity lost at each distance, from the gun; and their annexed differences show the quantity of velocity for each 20 yards or 60 feet, with its respective velocity.—The medium weight of the balls, in all the three days, being 16 oz 10·64 dr, or $16\frac{2}{3}$ oz, and diameter 1·965 inches.

Mean Velocities, &c, at several distances.						
Dists from pend	With 2 oz of powder			With 8 oz of powder		
	Velocity struck	Veloc. lost	Diffs.	Velocity struck	Veloc. lost	Diffs.
0	900	0		1637	0	
			34			74
60	866	34		1563	74	
			32			70
120	834	66		1493	144	
			30			66
180	804	96		1427	210	
			28			62
240	776	124		1365	272	
			26			59
300	750	150		1306	331	
			24			56
360	726	174		1250	387	

By employing here the theorem at Art. 131, for the resistances, we obtain the following table for the resistances to a ball of 1·965 inches diameter, and $16\frac{2}{3}$ oz weight.

With 2 oz of powder				With 8 oz of powder			
Velocity	Resistances			Velocity	Resistances		
feet	oz	diffs	2 diffs	feet	oz	diffs	2 diffs
883	261			1600	1030		
		25				101	
850	236		2	1528	929		9
		23				92	
819	213		3	1460	837		7
		20				85	
790	193		3	1396	752		8
		17				77	
763	176		1	1336	675		11
		16				66	
738	160			1278	609		

136. *August* 20, 1788; *from* 9 *till* 2.

Barometer 29·80; Thermometer 64.

Began this day a course of experiments with the gun n° 3, to fire it with a high charge, and several low ones, at different distances, to obtain the resistance to as many other velocities as can be got.—The pendulum was measured, weighed, and hung up two days ago, being a new block; but on using it, thought the wood not sound, as the balls that were fired, with 16 and 12 ounces of powder, passed quite through it, though it was 30¼ inches in length from front to back; the dimensions of its face were 22 inches deep, by 17¼ broad.—The weight at first when hung up was 824 lb; also the distance of its centre of gravity below the axis was 79 inches, and it vibrated 400 in 10 minutes.—The diameter of the balls measured from 1·96 to 1·97, so that the medium is nearly 1·965.—The distance of the gun to day was constantly 30 feet.

The balls of n°s 1, 3, 4, 5 went quite through the pendulum, and lodged in the bank behind it; so that the large charges were obliged to be discontinued. N° 19 just came out, and n° 22 was picked out again, as it was not far in; so that these, and all those that went through, or that rebounded back, are not added to the weight of the pendulum. The balls with 8 drams of powder, viz, n°s 22, 23, 24, 25, penetrated about 3 inches; and those with 4 drams, viz, 26, 27, 28, 29 penetrated about 1 inch, and rebounded back again, till they fell on the ground at about 12 feet before the face of the pendulum, which they struck at about the height of 4 feet; so that they must have rebounded from the pendulum with a velocity of about 24 feet per second, which of course is deducted from the velocities computed from the vibration of the pendulum at those numbers. The experiments here follow:

N°	Powder	Ball		Vibr. pend	Point struck	Plugs wt	Values of		Velocity of the ball	Mediums
							p	g		
	oz	oz	dr	c	inches	oz	lb	inches	feet	feet
1	16	16	11	188		4·0				
2	12	·	9	170	89·0	2·3	820·0	79·0	1837	
3	12	·	11	163	89·1	3·8				
4	12	·	12	141	87·8	2·7				
5	12	·	10	157	85·5	3·3				
6	4	·	11	124	90·1	2·9	821·1	79·0	1315	
7	4	·	6	120	85·7	3·9	822·2	·	1365	1346
8	4	·	12	121	85·6	3·2	823·3	·	1350	
9	4	·	10	122	86·9	2·9	824·4	·	1353	
10	3	·	12	108	87·9	2·9	825·5	·	1178	
11	3	·	11	110	87·2	3·1	826·6	79·1	1215	1185
12	3	·	9	113	86·1	3·0	827·7	·	D 1276	
13	3	·	9	106	88·7	3·3	828·8	·	1163	
14	1	·	9	57	89·6	2 ·7	829·9	·	D 620	
15	1	·	11	59	89·1	2·7	831·1	·	641	645
16	1	·	10	56	87·7	2·1	832·2	·	D 621	
17	1	·	14	60	87·5	1·3	833·3	·	658	
18	12dr	·	9	48	86·1	1·0	834·4	·	547	
19	12	·	13	51	86·5	2·0	835·5	·	569	546
20	12	·	9	47	88·4	1·0	835·5	·	522	
21	12	·	9	47	91·4	2·0	836·6	·	D 506	
22	8	·	12½	39	89·8		837·7	·	421	
23	8	·	10	43	93·7		837·7	·	450	449
24	8	·	8	44	93·1		838·8	·	467	
25	8	·	9	43	93·0		839·9	79·2	457	
26	4	·	10	27	78·9		841·0	·	318	
27	4	·	9	30	81·8		841·0	·	339	309
28	4	·	10	30	81·2		841·0	·	340	
29	4	·	11	30	80·7		841·0	·	341	

The medium weight among all these balls is 16 oz 10·2 dr.
The distance of the gun was 30 feet.

137. *August* 21, 1788; *from* 9 *till* 4.

Barometer 30·0; Thermometer 65. Continuation.

N°	Powder	Dist. gun	Vibra pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of the ball	Mediums
	oz dr	feet	c	inches	oz	lbs	inches	feet	feet
1	4	60	110	79·2	3·9	838·0	79·2	1363	1324
2	4	·	116	87·6	2·6	839·1	·	1306	
3	4	·	118	86·2	4·6	840·2	·	1323	
4	4	·	112	84·8	2·6	841·3	·	1301	
5	3	·	96	86·4	3·0	842·4	·	1088 D	
6	3	·	94	84·4	1·8	843·5	·	1092 D	
7	3	·	99	88·9	3·8	844·6	·	1102	1110
8	3	·	104	92·2	4·5	845 7	·	1118	
9	1	·	56	91·9	2·5	846·8	·	604 D	
10	1	·	52	88·6	2·6	847·9	·	583 D	
11	1	·	60	94·0	2·6	849·0	79·3	634	635
12	1	·	61	93·8	3·4	850·1	·	637	
13	dr 8	·	37	94·1		851·2	·	386 D	
14	8	·	39	91·9		852·3	·	421	423
15	8	·	38	91·6		853·4	·	414	
16	8	·	40	92·5		854·5	·	433	
17	4	·	28	90·4		855·6	·	290	287
18	4	·	23	82·7		855·6	·	D 259	
19	4	·	26	85·6		855·6	·	285	
20	4	·	25	87·1		855·6	·	D 269	
21	6	·	33	82·5		855·6	·	400	377½
22	6	·	33	83·6		856·7	·	376	
23	6	·	31	83·6		856·7	·	371	
24	6	·	31	84·3		857·8	·	363	
25	4 oz	120	99	78·1	2·6	858·9	79·4	1263	1248
26	4	·	108	87·9	3·3	860·0	·	1240	
27	4	·	108	87·9	2·2	861·1	·	1237	
28	4	·	111	89·5	3·0	862·2	·	1250	
29	3	·	94	89·7	3·0	863·3	·	1068	1072
30	3	·	89	85·4	3·3	864·4	·	1058	
31	3	·	91	87·1	3·4	865·5	·	1057	
32	3	·	101	92·5	0	866·6	·	1106	
33	1	·	51	84·7	1·9	864·3	·	607	613
34	1	·	54	86·1	3·1	865·4	·	628	
35	1	·	49	86·9	2·6	866·5	·	582	
36	1	·	52	88·5	2·5	867·7	·	603	

N°	Powder	Dist. gun	Vibr. pend	Point struck	Plugs wt	Values of p	Values of s	Velocity of the ball	Medium
	oz dr	feet	c	inches	oz	lb	inches	feet	feet
37	dr 8	120	37	99·1		868·7	79·4	400	
38	8	·	38	89·3		869·8	·	433	
39	8	·	30	85·4		870·9	79·5	D 356	419½
40	8	·	34	80·3		872·0	·	486	
41	8	·	33	81·5		873·1	·	409	
42	12	·	41	79·3		874·2	·	527	
43	12	·	44	87·0		875·3	·	524	523
44	12	·	47	87·2		876·4	·	555	
45	12	·	39	83·3		877·5	·	486	
46	6	·	30	87·3		878·6	·	354	
47	6	·	34	95·6		879·7	·	310	356
48	6	·	31	92·5		880·8	·	347	
49	6	·	35	87·0		881·9	·	414	
50	4	·	28	99·4		883·0	·	289	
51	4	·	28	95·1		883·0	·	287	277
52	4	·	27	84·4		883·0	·	D 306	
53	4	·	24	80·8		883·0	·	287	
54	12	60	41	79·4	1·3	883·0	·	534	
55	12	·	45	81·6		884·1	79·6	571	544
56	12	·	47	90·4		885·2	·	532	
57	12	·	47	90·4		886·3	·	538	

The balls at n°s 17, 18, 19, 20, 50, 51, 52, 53, mostly rebounded a little back.—The penetration with 6 drams of powder was about 3⅓ inches.—After the 32d number, 3 balls were taken out of the block.—N° 39 grazed, which reduced its velocity very much.—The pendulum now being full of balls, and quite spoiled, it was ordered to be taken down, and a new one to be prepared. Before taken down, it vibrated just 400½ in 10 minutes, or for each minute nearly 40·05 = n. When taken down, its centre of gravity measured 79·55 inches below the axis, and it weighed 887¼ lb. Now, in the two days,

The wt. of the balls lodged was -	66¾ lb	But at first it weighed	824 lb
And the wt. of plugs	7¾	Therefore it lost in the 4 days - -	11
Their sum is - -	74½	This loss is divided among all the days and rounds equally.	
Which taken from -	887¼		
Leaves only - - -	813		

The mean wt. of balls was 16 oz 11·05 dr.—The individuals 1, 2, or 3 drams over or under the mediums, as the former day. When the block was split up afterwards, to get out such balls as might be whole, it was discovered that the wood was quite sound, and very tough.

138. *September* 1, 1788; *from* 10 *till* 2.

Barometer 29·87; Thermometer, 62°.

The same experiments for the resistance of the air continued with a new pendulum. It was hung up 3 days before, and is 35 inches in length, from front to back. It then weighed 1014 lb; its centre of gravity was 80·8 inches below the axis; and it vibrated 439 in 11 minutes, or 39·91 per minute.

N°	Powder	Ball's wt	Dist gun	Vibr. pend.	Point struck	Plugs wt	Values of		Velocity of the ball	Mediums
							p	g		
	oz dr	oz dr	feet	c	inches	oz	lb	inches	feet	feet
1	4	16 15	180	85	83·4	2·8	1014·0	80·8	1216	
2	4	10	·	85	89·1	2·5	5·1	·	1160	1182
3	4	10½	·	89	92·5	2·3	6·4	·	1169	
4	4	10	·	82	86·9	2·7	7·4	·	1150 D	
5	3	14	·	75	86·5	2·7	8·6	·	1042	
6	3	12	·	75	84·7		9·7	·	1070	1053
7	3	8	·	74	86·5	3·0	20·9	·	1050	
8	3	11½	·	79	88·8	3·0	2·0	·	1083 D	
9	1	11	·	46	90·3	2·0	3·2	·	622	
10	1	12	·	41	81·7	0·5	4·3	·	611	
11	1	11	·	38	78·3		5·5	·	594	612
12	1	9½	·	40	79·1		6·6	·	623	
13	dr 12	11	·	37½	89·6		7·8	·	513	
14	12	11	·	39	92·9		8·9	80·9	516	
15	12	10	·	37	90·2		30·1	·	507	518
16	12	10	·	36	89·1		1·3	·	536	
17	8	10	·	33	92·4		2·4	·	442 D	
18	8	14	·	28	86·9	0·2	2·4	·	393	
19	8	13	·	30	90·4		3·6	·	407	407
20	8	12	·	32	93·5		4·7	·	421	
21	6	11	·	28	92·5		5·9	·	375	359
22	6	10	·	24	86·7		4·7	·	344	

Nos 17 and 22 rebounded gently; also 21 came out easily, and at the same time took out another.—The mean penetration, with 4 oz of powder, was 13 inches; with 3 oz was 10·9 inches; with 1 oz was 3·5 inches; with 12 dr was 2·8 inches; which penetrations are nearly proportional to the charges of powder.—Took out 2 lb 10 oz of broken balls at the end of this day's experiments.—The wood of this new pendulum proved to be very green and wet, so much so, that the water sprang and dropped out very fast, as the balls and plugs entered it. The medium weight of the balls was 16 oz 11·2 dr, with small variations, nearly as before.

139. *September* 2, 1788, *Experiments Continued.*

Barometer 30; Thermometer 65°.

No	Powder	Dist. gun	Vibr pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of ball	Mediums
	oz dr	feet	c	inches	oz	lb	inches	feet	feet
1	dr 6	180	26	87·0		1032·2	80·9	370	372
2	6	180	29¼	96·3		3·3	·	375	
3	4 oz	240	77	84·6	3·3	3·3	·	1120	1129
4	4	·	77	86·6	2·5	4·5	·	1115	
5	4	·	86	94·9	2·9	5·6	·	1117	
6	4	·	87	92·1	3·8	6·8	·	1165	
7	3	·	74	86·8	2·1	7·9	·	969	1008
8	3	·	70	86·7	3·7	9·1	·	928	
9	3	·	72	85·4	4·0	40·2	·	1056	
10	3	·	72	83·8	1·5	0·2	·	1079	
11	1	·	37	80 8	2·0	1·4	·	568	602
12	1	·	43	85·1		2·5	·	622	
13	1	·	39	80·0		3·7	·	608	
14	1	·	43	86·8		3·7	·	610	

140. *September* 3, 1788, *Experiments Continued.*

Barometer 30; Thermometer 65°.

No	Powder	Dist. gun	Vibr. pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of ball	Mediums
	oz	feet	c.	inches	oz	lb	inches	feet	feet
1	4	300	83	95·8	4·7	1044·8	80·9	1086	
2	4	·	76	87·3	3·2	6·0	·9	1076	1077
3	4	·	80	94·5	4·6	7·1	81·0	1069	
4	4	·	72	82·9	4·1	8·3	·	1077	
5	3	·	66	87·5	4·1	9·4	·	947	
6	3	·	72	94·8	5·5	50·6	·	951	956
7	3	·	68	81·7	4·0	1·7	·	970	
8	3	·	59	79·5	3·0	2·9	:	938 D	
9	4	360	70	83·6	5·4	4·0	·	1050	
10	4	·	71	87·9	6·5	5·2	·	D 1008	1034
11	4	·	79	95·0	6·1	6·3	·	1040	
12	4	·	75	91·7	4·4	7·5	·	1040	
13	3	·	63	88·9	3·4	8·6	·	894	
14	3	·	60	81·1	4·5	9·8	·	954	936
15	3	·	63	85·0	3·4	60·9	·	934	
16	3	·	65	87 5	3·2	2·1	·	962	

Nos 12 and 14 grazed very lightly before they struck the pendulum. And yesterday took out 3 balls, viz, nos 2, 9, 13.—After the experiments this day, both the front and back of the pendulum were covered with sheet lead, each piece being in size 22¼ inches by 17¼; and the two together weighed 41 lb.—The mean weight of the balls these two days, was 16 oz 11$\frac{7}{16}$ dr, with small variations, as before.

141. *Sept.* 4, 1788, *from* 10 *till* 3; *Experiments Continued.*

Barometer 29·82; Thermometer 74°.

N°	Powder	Dist. gun	Vibrat pend.	Point struck	Plugs wt	Values of p	Values of g	Velocity of ball	Mediums
	oz	feet	c	inches	oz	lb	inches	feet	feet
1	16	360	92	82·8		1104·2	81·2	1483	
2	16	·	105	83·1	5·2	5·4	·	1681	1582
3	16	·	97	90·0	6·3	6·5	·	D 1425	
4	16	·	97	91·0	6·9	7·7	·	D 1416	
5	12	·	89	80·5	4·8	8·8	·	1476	
6	12	·	107	95·0	9·5	10·0	·	1512	1492
7	12	·	105	93·7	6·5	1·1	·	1487	
8	12	·	95	90·6	8·9	2·3	·	D 1414	
9	16	300	109	89·9	11·7	3·4	·	1613	
10	16	·	99	85·4	4·0	4·6	·	D 1522	1648
11	16	·	102	83·6	7·8	5·7	·	1645	
12	16	·	104	81·7	11·0	6·9	·	1685	
13	12	·	101	82·3	9·1	8·0	·	1652	
14	12	·	75			9·2	·		
15	12	·	110	92·0	5·8	20·3	·	1584	1572
16	12	·	92	81·5	6·9	1·5	·	1530	
17	12	·	93	82·3	5·2	2·6	·	1521	
18	16	240	106	81·4	8·3	3·8	·	1781	
19	16	·	111	91·4	7·0	4·9	·	1645	
20	16	·	92D	89·2	8·4	6·1	·	D 1404	1713
21	16	·	86D	82·9	8·8	7·2	·	D 1397	
22	16	·	92D	81·4	9·9	8·4	·	D 1524	

The ball n° 14 grazed the screen. At n° 12 the vent of the gun began to run or melt, and was much widened at the end.—The last three numbers are very doubtful, and evidently too small, perhaps owing to the balls striking other balls within the block, and changing their direction there. The mean weight of the balls was 16 oz 10·86 dr.—After these experiments, the pendulum oscillated 399 in 10 minutes: being now full of balls, and much rent and shattered, it was taken down, to be replaced, and balanced and weighed, when its weight was found to be 1130 lb, and its centre of gravity was 81·2 below the axis.—The number of balls that had struck the pendulum, was 74, and

Their whole weight was -	77 lb ; of these there	
Rebounded or took out $9\frac{1}{2}$ =	10	
Wt. of balls remaining -	67	
Wt. of plugs added - -	8	
Wt. of lead added - - -	41	
Whole weight added - -	116	
Wt. at first - - - - - -	1014	
Wt. at last - - - - - -	1130	

142. *September* 15, 1788 ; *from* 10 *till* $3\frac{1}{2}$.
Barometer 30·09 ; Thermometer 60°.

N°	Powder	Dist. gun	Vibra. pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of the ball	Mediums
	oz	feet	c	inches	oz	lbs	inches	feet	feet
1	16	30	100	89·6	10	1370	81·1	D 1855	
2	·	·	112	88·5	8	1	·	2100	
3	·	·	118	90·6	8	3	·	2143	
4	·	·	115	87·7	9	4	·	2202	2088
5	·	·	108	84·8	14	6	·	2133	
6	·	·	105	83·8	17	7	·	2093	
7	·	60	92	80·4	12	9	·	D 1929	
8	·	·	98	81·9	7	80	·	2040	
9	·	·	106	89·8	16	2	81·2	2010	1974
10	·	·	105	91·9	14	3	·	D 1920	
11	·	·	107	91·9	15	5	·	1973	
12	·	120	103	91·0	14	6	·	1899	
13	·	·	102	83·4	14	8	·	D 2071	
14	·	·	98	86·1	16	9	·	1935	
15	·	·			8	91	·		1896
16	·	·	96	88·4	16	2	·	1839	
17	·	·	97	85·5	9	4	·	1913	
18	·	180	95	88·1	8	5	·	1825	
19	·	·	88	90·9	13	7	·	D 1643	
20	·	·	101	91·2	14	8	·	1855	
21	·	·	105	90·1	15	1400	·	1953	
22	·	·	92	94·5	11	1	·	D 1653	1781
23	·	·	87	85·7	11	3	81·3	1745	
24	·	·	77	87·7	10	4	·	D 1507	
25	·	·	88	84·8	10	6	·	1792	
26	·	240	92	88·0	13	7	·	1808	
27	·	·	88	91·4	11	9	·	1643	
28	·	·	95	89·2	7	10	·	1819	1741
29	·	·	83	87·8	9	2	·	1623	
30	·	·	96	89·6	7	3	·	1815	

These experiments were performed with a new pendulum, of 50 inches in length, from front to back, consisting of two pieces of elm firmly jointed and bolted together, that its length might bear the great charges of 16 oz of powder. It was hung up two days before, when it weighed 12 cwt 1 qr 5 lb, or 1377 lb, and the distance to its centre of gravity was 81·3 inches. It oscillated this morning 395 in 10 minutes, or 39·5 per minute, and at the end of the experiments it oscillated the very same; but its centre of gravity was now found to be only at 81·1, being 0·2 less than before, whereas it ought to have been the contrary. In this doubtful case, I have taken 81·2 for the middle, with 0·1 less for the beginning, and 0·1 more for the end.—The number of balls lodged is 30, and their weight is $31\frac{1}{4}$ lb, which is on an average at the rate of 16 oz 10·7 dr each; their mean diameter 1·96. Now,

Weight of balls lodged - -	$31\frac{1}{4}$ lb
Weight of plugs - - - - -	$21\frac{1}{2}$
Weight of pendulum at first	1377
The sum is - - - - -	$1429\frac{3}{4}$
Weight at the end - - -	1413
The difference is - - - - -	$16\frac{3}{4}$

This great loss of $16\frac{3}{4}$ lb, I am at a loss fully to account for, especially as there was no loss at all in the preceding pendulum. The wood was very green and wet to be sure, and the pendulum hung up for 3 days before it was weighed; besides, some small pieces flew out by the stroke of the balls, as well as some of the facing of sheet lead. So that there is reason to suspect an error of a few pounds had taken place on the first weighing. On all these accounts I have made the weight at the end of the experiments to be 1415 lb, and at the beginning of them 1370 lb, and divided the difference, 45 lb, equally among the 30 rounds.

The medium weight among all the balls, in the 7 days experiments, has been 16 oz 10·88 dr = 16·68 oz. Now the diameter answering to this, found from the proportion of

the 18 oz ball being 2 inches, or the 9 lb ball being 4 inches, is 1·95; for $\sqrt{15} : \sqrt{16{\cdot}68} :: 2 : 1{\cdot}95$. But, by measurement, we commonly concluded the mean diameter to be nearly 1·96 or 1·965, a very little more or less.

143. *September* 25, 1788. *Barom.* 30; *Therm.* 57°.

Experiments with 3 pounder guns.

No	Powder	Ball's wt		Dist. gun	Vibr. pend	Point struck	Penetrat.	Values of		Velocity of ball
								p	g	
	oz	oz	dr	feet	c	inches	inches	lb	inch	feet
1	4	49	4	60	101	83·8	7¼	1421·0	82·10	710
2	4	•	•	•	125	95·5	9¼	24·5	·12	773
3	4	•	•	•	103	87·7	8	27·9	·14	695
4	4	•	•	•	117	90·1	10	31·4	·16	771
5	4	•	•	•	101	83·5	7	34·8	·17	720
6	4	•	•	•	121	90·5	11¼	38·3	·19	798
7	16	•	•	•	243	90·1		41·8	·21	1614
8	16	•	•	•	191	84·6		45·2	·22	1354
9	16	•	•	•	234	89·1		48·7	·24	1580
10	16	•	•	•	197	88·6		52·1	·26	1341
	16	•	•	•	238	91·4		55·6	·28	1574
11	16	•	•	•	180	83·9		59·1	·29	1300

These experiments were made with two 3-pounder brass guns, the one very long, and the other short, fired alternately, first the one and then the other, and each of them both with 4 oz and 16 oz of powder; to find the effect both of length of gun, and weight of powder. The balls were all exactly of an equal weight and diameter, viz,

Weight of balls - - - - - - - - 3 lb 1 oz 4 dr
Diameter of ditto - - - - - - 2·84 inches
Length of bore of long gun - - - 69 inches
Ditto of short gun - - - - - - 34½ inches
Distance of pendulum - - - - - 60 feet
Weight of pendulum 6 days before - 1426 lb
Centre of gravity then, at - - - - 81·75 inches
Oscillated then 397 in 10 minutes
Wt. of pendulum after the exper. - 1462½ lb
Centre of gravity then at - - - - 82·63
And oscillated 396 in 10 minutes.

Mediums of Velocities.

Short Gun		Long Gun	
4 oz	16 oz	4 oz	16 oz
708	1332	781	1589

Weight of pendulum 6 days before -	1426 lb
Weight of balls lodged - - - -	37
Weight of plugs - - - - - - - -	$4\frac{1}{2}$
Sum - - - - - - - - - - - -	$1467\frac{1}{2}$
Weight of pendulum at the end - -	$1462\frac{1}{2}$
Weight lost in the 6 days - - - -	5

As the weight, $41\frac{1}{2}$ lb, of the balls and plugs, will only change the centre of gravity $\frac{2}{10}$ inch, some small error must have been made in the two measurements of that centre; I have therefore taken a mean between them.—It appears that the velocities, with each, are nearly as the square roots of the charges, but more nearly in the long gun. The long gun has the advantage by about $\frac{1}{10}$ in the small charge, and by about $\frac{1}{5}$ in the high charge; which advantages are again as the velocities, or as the roots of the charges.—The mean penetrations, with the 4 oz charges, are $7\frac{1}{2}$ and 10 inches, or nearly as 2 to 3, which are nearly as the 4th power of the velocities, and also nearly as the square roots of the lengths of the guns.

144. It will now be proper to collect a summary of the preceding seven days experiments, with the 1-pound balls, and the gun n° 3. Now these experiments have been made with the following nine varieties of charges of powder, viz, 16 oz, 12 oz, 4 oz, 3 oz, 1 oz, 12 dr, 8 dr, 6 dr, 4 dr; the charges of 8 oz and 2 oz having been omitted this year, as these were pretty amply made in the preceding year 1787. The medium velocities, for each of these charges, with which the balls struck the pendulums, placed at the several distances of 30, 60, 120, 180, 240, 300, 360 feet, when collected and ranged in columns under their several charges, and on the lines of their respective distances, will be as here exhibited in the following table.

Table of Experimented Medium Velocities.

Dists. feet	16 oz	12 oz	4 oz	3 oz	1 oz	12 dr	8 dr	6 dr	4 dr
30	2088	1837	1346	1185	645	546	449		309
60	1974		1324	1110	635	544	423	378	287
120	1896		1248	1072	613	523	419	356	277
180	1781		1182	1053	612	518	407	359	
240	1727		1129	1008	602				
300	1648	1572	1077	956					
360	1582	1492	1034	936					

Now, there being evidently, and indeed unavoidably, certain small irregularities among these numbers, in each column, by making among them some small corrections, by adding or subtracting, as the terms may appear to vary from a general law of progression; and also reducing the series to commence at 0 instead of 30 feet, that the distances may all have the same common difference 60; then the more regular mediums will be as below:

Table of the Regular Mean Velocities.

Dists	16 oz	12 oz	4 oz	3 oz	1 oz	12dr	8 dr	6 dr	4 dr
0	2100	1984	1360	1203	662	557	450		302
60	2005		1300	1151	641	543	437	381	292
120	1914		1243	1102	622	530	425	370	283
180	1827		1189	1056	606	518	414	360	
240	1744		1138	1013	590				
300	1665	1569	1090	973					
360	1590	1499	1045	936					

Then by applying the theorem in Art. 131, viz, $r = \frac{vv'}{32}b = \frac{vv'}{1920}b$, for resistances, to these numbers, we shall obtain the following table of resistances, in ounces, to the corresponding annexed velocities in feet, as derived from the preceding experiments with the several charges of powder, viz, 16 oz, 8 oz, 4 oz, 2 oz, 1 oz, 12 dr, 8 dr, 6 dr, 4 dr, reduced and adapted to a ball of 2 inches diameter, and to 30 inches of the barometer.

	Velocity	Resist	Diffs.	2 Diffs
	feet	oz		
with 4 dr	287	22	8	
	297	25		
6 dr	365	32	4	
	376	36		
8 dr	420	40	5	1
	432	45	6	
	444	51		
12 dr	524	55	6	0
	537	61	6	
	550	67		
1 oz	598	78	13	1
	614	91	14	1
	632	105	15	
	652	120		
2 oz	738	160	16	2
	763	176	18	2
	790	194	20	2
	819	214	22	3
	850	236	25	
	883	261		
3 oz	955	306	38	4
	993	344	42	3
	1035	386	45	4
	1079	431	49	3
	1127	480	52	
	1177	532		
4 oz	1068	418	47	4
	1114	465	51	4
	1164	516	55	4
	1216	571	59	4
	1272	630	63	
	1330	693		
8 oz	1278	609	68	8
	1336	677	76	8
	1396	753	84	8
	1460	837	92	9
	1528	929	101	
	1600	1030		
16 oz	1627	1060	110	8
	1705	1170	118	9
	1786	1288	127	8
	1872	1415	145	9
	1960	1550	134	
	2053	1694		

These numbers, though not perfectly regular, must at least be very near the truth, as appears by the uniformity of the two orders of differences, which are quite sufficiently regular, in each series of the numbers with the same charge of powder, and leaving only some doubt about the point of uniformity in passing from the series with one charge to that for the next charge; which doubt will be cleared up, and the deviations from regularity corrected afterwards, when we come to construct the curve of resistance, from the numbers furnished by this table. The number of these terms might have been greatly multiplied, by taking many intermediate terms, and thence computing the resistances at much smaller intervals of velocities; but the annexed number will be found quite sufficient, being every one of those computed for the medium velocities at the middle of every 60 feet of distance, between the successive intervals 0, 60, 120, 180, 240, 300, 360.

It was not found practicable to carry on the experiments at distances beyond the 360 feet, on account of the uncertainty of hitting the pendulum properly so far off, very many of the balls missing entirely, or striking it towards the edges, there breaking the iron bands, and demolishing the apparatus. Neither was it practicable to employ, in this way, less

charges than that of 4 drams of powder; because with a less velocity, the balls would not lodge in the wood, but made a small perforation, and rebounded back again. However, the resistances to all the smaller velocities we shall be afterwards able to supply, from the experiments that we have made with Mr. Robins's Whirling Machine, hereafter to be related; by which means we shall find the two kinds of experiments uniting, and forming between them one general and uniform series. Besides the few small irregularities above mentioned, some slight deviation from perfect accuracy may possibly arise from employing for the velocity v, in the theorem $r = \frac{vv'}{1920}b$, the mean between the extremes, or first and last velocities, at every 60 feet distance, for the constant velocity on which the resistance is made, as due to that velocity.

145. But now to examine what degree of approach to uniformity the foregoing series of resistances possesses, let us construct the curve, called the Curve of Resistance, taking for the abscisses, the numbers denoting the resistances, and for the ordinates the correspondent velocities. Thus, in plate 2, draw the two indefinite lines AB, AC, perpendicular to each other. From any scale of equal parts, set off from A, upon AB, the series of velocities of the first column, increasing from the top or smallest, to the bottom or largest. In like manner, set the correspondent resistances upon the line AC downwards. Then from every pair of these terminating points, draw lines parallel to AB and AC, and the intersections of these will be so many points in the curve of resistance; through which the curve may be drawn with a steady hand, or by a thin pliant slip of card paper bent round by the points; then the regular curve is also easily drawn, as is here represented by the continued black curve line AD, from which the experimented curve differs only in two or three parts, by the smallest perceptible variations.

Now, from this regular curve, by measurement, we can easily deduce a more accurate series of numbers, for the

resistances, than the experimented one, adapted to any uniform series of velocities. Thus, to adapt a series of resistances to a series of velocities for every 100 or every 50 feet of difference: dividing the line AB into equal spaces of 50 feet, and from the points of division drawing lines parallel to AC, cutting the curve in as many points E; then the perpendicular distance of these points E, from the line AB, will be the correspondent resistance EF; or drawing EG parallel to AB, then AG will be equal to the same resistance, due to the velocity AF or GE. Thus, by a nice measurement, and altering the numbers here-and-there by only a unit or two, is obtained the annexed table of resistances, for a ball of 2 inches diameter, on every 100 feet of velocity, from 300 to 2000 feet velocity.

Velocity	Resist.	Diffs.	2 Diffs
feet	oz		
300	25·8		
		20·7	
400	46·5		7·2
		27·9	
500	74·4		8·2
		36·1	
600	110·5		9·4
		45·5	
700	156·0		10·5
		56·0	
800	212·0		12·3
		68·3	
900	280·3		13·4
		81·7	
1000	362·0		13·2
		94·9	
1100	456·9		12·6
		107·5	
1200	564·4		11·4
		118·9	
1300	683·3		9·3
		128·2	
1400	811·5		7·4
		135·6	
1500	947·1		4·2
		139·8	
1600	1086·9		1·7
		141·5	
1700	1228·4		-1·3
		140·2	
1800	1368·6		-3·1
		137·1	
1900	1505·7		-5·0
		132·1	
2000	1637·8		

From this table, the resistance due to any given velocity, or the velocity due to any given resistance, may easily be derived, either by inspection or interpolation, as well for any other ball, as for this ball, of 2 inches diameter, by stating the resistance to be as the square of the ball's diameter, as it is nearly. Many other properties and uses may be derived from this curve and table of experiments, which it will be more proper to advert to after the remaining experiments have been stated.

OF THE EXPERIMENTS IN 1789.

146. The experiments projected for this year, were of various kinds, with cannon and mortars, both with the ballistic pendulum, and discharged at elevations for the ranges.

The few 3-pound balls that were discharged at the ballistic pendulum, at the conclusion of the last year's experiments, gave reason to hope that the pendulum might be able to bear that size of balls, and that, by enlarging its dimensions, it might even bear the shock of balls of a still larger size: it was therefore resolved, that the experiments of 1789 be prosecuted with balls of 3 lb weight and upwards. Accordingly two very large pendulum blocks were prepared, and a proportionally strong apparatus of iron to bind each firmly together, and to suspend it, the whole amounting to almost a ton weight.

Two guns were also prepared, a long and a short 3-pounder, of exactly the same calibre, and the bottom of the bores also alike, being both a hemispherical cavity. The diameter of each bore was 2·94 inches, the length of the longer being 69¼ inches, and of the shorter 40 inches.

The chief object was to obtain the measure of the air's resistance, which it was proposed to determine by discharging the balls at the pendulum from different distances, observing how much velocity was lost by moving through so many feet of air, and also by projecting them at elevations, to fall in the river for the corresponding ranges. Several other collateral objects were also attended to in the progress of the experiments.

The states of the barometer and thermometer were taken every day, as usual, to judge of the effects of the weather on the powder, and on the resistance of the air to the balls. The powder, in the experiments, was put into thin flannel cartridges as usual; and it was first proved every day by

my new eprouvette, which is so correct as to show any very minute differences in its strength.

147. *October* 13 *and* 14, 1789.

Employed these two days in weighing, measuring, balancing, and suspending the first new pendulum. It was of sound, dry elm; and its dimensions were, 54 inches in length from front to back, and the face or end 24 inches deep, by 18 inches broad: the balls to be fired as usual in at the ends, in the direction of the fibres, first one end, and then the other. Its weight, with the spear, was 1655 lb; the centre of gravity at 77·3 inches below the middle of the axis of suspension. And it oscillated, in small arcs, 399 times in 10 minutes; which gives $n = 39·9$ or 40 nearly.

148. *October* 19, 1789. *Cloudy air.*

Barometer 29·64; Thermometer—

Fired only 6 rounds this day, viz, 3 with each gun, first the short gun, and then the long one; the muzzle of each being at 30 feet distance from the face of the pendulum. Every charge of powder was 16 oz; the weight of every ball was 2 lb 15 oz, and their diameter 2·78 inches.

N°	Vibr. pend	Point struck	Plugs wt	Penet ball	Values of		Veloc. Mediums
					p	g	
	e	inches	oz	inches	lb	inches	feet.
1	184	91·0		23·6	1655	77·3	
2	181	89·4	10		58	·3	1371
3	178	86·3	10		60	·3	
4	212	89·7	7		63	·4	
5	200	86·9	5	29·3	66	·4	1584
6	201	85·7	4½		69	·4	

149. *Oct.* 20, 1789. *Fine weather.*

Barometer 29·82.

The first 3 rounds this day were made with the short gun, firmly blocked up behind, to prevent its recoil entirely, to try if that would make any difference in the velocity. The remainder were made with the long gun, at different distances. And in all other experiments the gun was permitted to recoil. Three trials with the eprouvette gave severally 30, 35, 32, the medium of which is 32¼. These were three discharges with my new eprouvette, (to be described afterwards) used here and in the following experiments, to show the uniformity of the powder, 2 oz being fired each time, put into flannel bags, at least till otherwise noticed. The divisions are from the scale of chords, marked *velocity*, which shows the proportional strength of the powder, when fired without a ball. The balls, in all the fifteen rounds, the same as before, viz. weight 2lb 15oz, and 2·78 inches diameter on a medium; the charge of powder always 16 oz.

N°	Dist. pend.	Vibr pend	Point struck	Wt Plug	Penetr Ball	Values of		Velocity mediums
						p	*g*	
	feet	c	inches	oz	inches	lb	inches	feet
1	30	165	83·2	7	30·3	1671	77·4	
2	•	179	87·9	5		74	·5	1353
3	•	174	89·4	8		77	·5	
4	80	182	80·7	7	30·5	79	·5	
5	•	168	78·9	4		82	·5	1490
6	•	196	91·9	6	29·5	85	·5	
7	130	186	83·2	8		87	·6	
8	•	189	88·0	7		90	·6	1487
9	•	202	94·3	9	27·0	93	·6	
10	180	186	90·0	8		96	·6	
11	•	188	90·7	8		98	·6	1422
12	•	183	89·0	8		1701	·7	
13	230	176	86·5	7		4	·7	
14	•	165	83·2	7		6	·7	1393
15	•	178	88·5	8		9	·7	

150. *October* 21, 1789. *Fine weather.*

Barometer 39·55. Long Gun.

Eprouvette 30, 32, 35, medium 32⅓. The powder in bags. The charge 16 oz. The balls the same as before.

N°	Dist pend	Vibrat pend	Point struck	Wt plug	Penetr ball	Values of p	Values of g	Veloc mediums
	feet	c	inches	oz		lb	inches	feet
1	330	159	84·0	9		1712	77·8	
2	·	165 D	93·0	9	grazed	15	·8	1330
3	·	174	92·5	9		17	·8	
4	·	158	84·6	9		20	·8	
5	430	168	94·9	7		23	·8	
6	·	154	88·6	8		25	·9	1227
7	·	172	96·3	9		28	·9	D

151. *October* 22, 1789. *Cold and Clear.*

Barometer 29·97; Thermometer 58.

The long gun. Medium of eprouvette 30, powder in bags. Charge of powder 8 oz, and balls as before.

N°	Dist pend	Vibrat pend	Point struck	Wt plug	Penetr ball	Values of p	Values of g	Veloc mediums
	feet	c	inches	oz	inches	lb	inches	feet
1	30	133	86·2	4		1731	77·9	
2	·	135	89·5	4	16·0	34	·9	1062
3	·	132	86·3	5		36	·9	
4	80	124	83·7	6		39	78·0	
5	·	127	85·8	4		42	·0	1043
6	·	138	89·3	5		44	·0	
7	130	135	92·8	5	15·8	47	·0	
8	·	140	97·0	5		50	·1	1016
9	·	128	89·1	6		52	·1	
10	180	119	84·9	5		55	·1	
11	·	121	82·8	4		58	·1	1016
12	·	128	88·3	5		61	·1	

152. *October* 23, 1789. *Fine weather.*

Barom. 30; Thermom. 58.

Long gun. Medium of eprouvette 30·8, with the powder in bags. The charge of powder 8 oz. Balls wt 2lb 15oz, diameter 2·78.

N°	Dist pend	Vibra pend	Point struck	Wt plug	Penetr ball	Values of		Veloc mediums
						p	*g*	
	feet	c	inches	oz	inches	lb	inches	feet
1	230	115	83·5	5	16·5	1763	78·2	
2	·	112	83·8	4		66	·2	960
3	·	105	78·5	4	15·4	69	·2	
4	330	110	86·6	4		71	·2	
5	·	107	85·2	4		74	·2	924
6	·	104	77·6	7		77	·3	
7	430	110	87·8	5		80	·3	
8	·	102	84·3	4		82	·3	902
9	·	102	82·3	9		85	·3	

153. *October* 23, *continued.*

The following rounds were made with a 6-pounder gun. The ball's diameter 3·48, and weight 5lb 13oz; gun's distance 80 feet.

N°	Powder	Vibra pend	Point struck	Wt plug	Values of		Veloc mediums
					p	*g*	
	oz	c	inches	oz	lb	inches	feet
1	8	157	86·6		1788	78·4	
2	·	169	88·4	7	93	·4	703
3	·	158	87·6		99	·4	
4	16	201	85·9		1804	·5	990
5	·	260	94·8	9	09	·5	
6	·	262	92·8	9	15	·6	
7	24	295	90·3	9	20	·6	1222
8	·	290	94·0	9	26	·7	

154. *October* 26, 28, 1789.

The pendulum now oscillated 401 times in 10 minutes, which gives the value of $n = 40{\cdot}1$, or nearly 40. Being quite spoiled, it was taken down, and found to weigh 1831lb. Now, the

	lb	oz
Wt of balls lodged is	191	11
Wt of plugs ditto	21	14
Wt of pend. at first	1655	0
The sum is	1868	9
Wt now found, only	1831	0
Thereof lost by evaporation and splinters	37	9

1831lb weight now,
1655lb weight at first

65) 176 (2·7 is the mean increase of weight to be added for every round with the 3-pounder, and double the same with the 6-pounder, to give the several values of p.

Now, for the values of g: because 5985·3 is the sum of the distances of the points struck by the balls, below the axis, and 65 the number of them, counting the 6-pounders twice; therefore

65) 5985·3 (92·1 gives the mean point struck,
and 77·3 is the centre of gravity at first

therefore 1831 : 176 : : 14·8 : 1·4 change in the centre of gravity
77·3 its place at first

78·7 the new place of ditto.

Also 65) 1·4 (·02154 is the mean change in the value of g for every 3-pound ball, and double the same number for each 6-pound ball.

From these data, the several values of p, g, &c, being made out, and the medium velocities computed, these and the mediums of the balls, &c, will be as in the following table.

N°	Dist gun	Powder	Ball's diam	Ball's wt	Veloc. ball	Velocities all reduced to a ball of 3lb wt. and 2·78 diam. & barom. 30.	
	feet	oz	inches	lb	feet	feet	
1	30	16	2·78	2·938	1371	1357	short gun
2	30	·	·	2·938	1353	1339	short gun
3	30	·	·	2·938	1584	1568	long gun
4	80	·	·	2·933	1490	1475	long gun
5	130	·	2·79	2·958	1437	1471	long gun
6	180	·	2·78	2·953	1422	1410	long gun
7	230	·	·	2·964	1393	1384	long gun
8	330	·	·	2·922	1330	1310	long gun
9	430	·	·	2·984	1227	D1219	long gun
10	30	8	·	3·000	1062	1061	long gun
11	80	·	·	3·000	1049	1048	long gun
12	130	·	·	3·000	1016	1015	long gun
13	180	·	·	3·000	1016	1015	long gun
14	230	·	·	3·000	960	960	long gun
15	330	·	·	2·985	924	922	long gun
16	430	·	·	2·953	902	895	long gun
17	80	8	3·50	5·823	703		6-pounder
18	80	16	3·50	5·854	990		6-pounder
19	80	24	3·475	5·750	1222		6-pounder

Here the velocities are reduced, in the last column, to what they would have been, had the barometer been always 30 inches, and if the balls had been all of just 3lb weight, by diminishing them according to the inverse ratio of the square roots of the weights, and of the diameters of the balls.

Some Experiments with the Gun, and some Mortars, discharged at Elevations down the River, to find the Ranges.

155. *October 26.*

Two iron mortars, of 8 and 10 inches, were fired, both with shells and solid balls, their fall in the water being observed from the opposite sides, with two theodolites, by lieutenants Wm. Mudge and Wm. Dixon.

Diameter of smaller bore 7·875,
Ditto of larger . . . 9·875.

N°	Mortar	Elevation	Powder	Ball or Shell diam	Ball or Shell wt	Time flight	Range	
	inches		lb oz	inches	lb	sec	yards	
1	8	60°	2 4⅞	7·69	64¼	25	1500	Solid balls
2	8	·	· ·	7·68	63¾	24	1400	
3	8	·	· ·	7·69	48	26½	1675	Shells
4	8	·	· ·	7·68	48	27	1625	
5	10	·	4 8	9·74	129	25½	1850	Solid balls
6	10	·	· ·	9·74	127½	25½	1635	
7	10	·	· ·	9·74	96	27	1770	Shells.
8	10		· ·	9·74	96		2075	

156. *October* 27, 1789. *Barometer* 30.

The long 3-pounder Gun.

N°	Elevation	Powder	Ball's wt	Ball's diam	Time	Range	Angle of fall
			lb oz	inches	sec	yards	
1	45°	8	2 15	2·78		2420	These ranges very doubtful.
2	·	8	· ·	·		2400	
3	·	8	· ·	·		3270	
4	·	4	· ·	·		3160	
5	·	4	· ·	2·70	23		
6	·	2	· ·	·	16	1290	
7	·	4	· ·	2·78	22½	1950	
8	30	8	· ·	·	22	2560	
9	20	8	· ·	·	12	1600	

157. *October* 28, 1789.

The 3-pounder gun continued.

N°	Elevation	Powder	Ball's wt	Ball's diam	Time	Range	Angle of fall
			lb oz	inches	sec	yards	
1	45°	8	3 0	2·78		2450	
2	·	·	· ·	·		2420	
3	·	·	· ·	·		2330	
4	·	·	· ·	·		2220	
5	·	16	· ·	·		2640	61°
6	·	·	· ·	·		2720	
7	·	·	2 15¾	·		2640	
8	·	·	· ·	·		2840	61½
9	·	24	2 15½	·		2640	

158.

Oct. 28. *The* 8-*inch Mortar.* The Charge 24 oz.				*Nov.* 10. *The* 8-*inch Mortar.* Charge 32oz. Elevation 45°.		*Nov.* 10. 3-*lb Gun.* Elevation 75°.		
N°	Eleva	Fall	Range	N°	Range	N°	Powder	Range
			yards		yards		oz	yards
1	35°	58°	1900	1	1920	1	8	1920
2	35	51	2200	2	2020	2	8	2050
3	55	61½	1700	3	2120	3	16	1890
				4	1960	4	·	2070
				5	2120	5	·	1270
				6	1930	6	24	1870
				7	2070	7	·	2300
				8	2080	8	·	1740

THE PENDULUM EXPERIMENTS RESUMED.

159. *Nov.* 14, 1789.

A second new pendulum having been prepared, it was this day weighed, measured, balanced, &c, as below.

Dimensions: length 50¾ inches, breadth 18¼, depth 23½. Its weight 1748lb; its centre of gravity on the first end at 79¼, at the second end 79·3. When hung up, and oscillated, it vibrated 397 in 10 minutes; which gives $n=39·7$.

160. *November* 16.

Barometer 29·4; Thermometer 51.

The following balls were discharged at the pendulum, from different distances, for the resistance of the air, the first 10 from the long 6-pounder gun, the latter 7 from a 4-pounder; the former with our usual powder, 2 oz of which gave a medium of 36¼ on my eprouvette; and the latter 7 with a stronger powder, which gave a medium of 38⅛ on the same eprouvette, as here below.

EPROUVETTE.

Old powder.	New powder.
36, $36\frac{2}{3}$, $36\frac{9}{10}$ } $36\frac{1}{2}$ mean.	38, $38\frac{1}{3}$, $38\frac{1}{8}$ } $38\frac{1}{6}$ mean.

The powder was put in the eprouvette loose, or without a bag, only set up neatly in the bottom of the bore; and it is remarkable, not only how much more uniformly it now acts, but also much more strongly, in the ratio of 36 to 30 or 31. The same method of charging the eprouvette is intended to be always practised in future. The diameter of the 3lb balls 2·78, of the 4lbs were 3·09 inches. Charge of powder with the 3lb gun was 24oz; with the 4lb it was $21\frac{3}{8}$ ounces.

N°	Dist gun	Ball's wt	Vibr. pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of the ball	Velocity reduced
	feet	lb oz	e	inches	oz	lb	inches	feet	feet
1	30	3 0	215	89·3	6	1748	79·3		
2	·	· ·	217	90·7	6	51	3	1727	1722
3	·	· ·	220	91·0	10	54	3		
4	80	· ·	214	89·8	9	56	3		
5	·	· ·	211	90·7	9	59	4		
6	·	· ·	195	91·6	9	62	4	1700	1695
7	·	· ·	214	90·2	10	65	4		
8	130	· ·	192	84·2	6	67	4		
9	·	· ·	203	91·0	9	70	4	1609	1604
10	·	· ·	190	88·7	8	73	4		
11	90	4 $1\frac{1}{2}$	187D	79·5	9	76	4		
12	·	· ·	113D	79·5	0	79	5	grazed	
13	·	· ·	184D	90·0	11	81	5		
14	·	· ·	212	89·0	10	84	5		
15	·	· ·	231	90·8	9	87	5	1268	1230
16	·	· ·	213	94·6	9	90	5		
17	·	· ·	203	90·8	10	92	5		

161. *November* 17. *Three-pounder continued.*

Barometer 29·96; Thermometer 51.

Powder 24 ounces.

No	Dist. gun	Ball's wt		Vibr. pend	Point struck	Plugs wt	Values of		Velocity of ball	Veloc reduced
							p	*g*		
	feet	lb	oz	*c*	inches	oz	lb	inches	feet	feet
1	180	3	0	203	91·5	5	1795	5		
2	•	•	•	189	91·5	6	98	6	1586	1578
3	•	•	•	188	87·3	10	1801	6		
4	230	•	•	171	79·7	9	3	6		
5	•	•	•	187	89·6	10	6	6		
6	•	2	15½	183D	93·3	10	9	6	1550	1546
7	•	•	•	185	91·5	10	12	6		
8	330	•	•	132D	90·3	5	15	6		
9	•	•	•	168D	91·2	9	17	7	grazed	
10	•	•	•	118D	88·1	10	20	7		
11	•	•	•	184	90·4	5	23	7		
12	•	2	15	169	84·5	8	26	7	1520	1498
13	•	•	•	165	85·8	9	28	7		
14	430	•	•	169	95·4	10	31	7		
15	•	•	•	164	96·4	10	34	7	1323	1300
16	•	•	•	141	82·8	10	37	7	grazed	

Eprouvette, 35¾, 36½, 36¾, the medium 36⅓.

The last column contains the velocities reduced to what they would be for a 3lb ball, with the barometer at 30, and the powder at 36 of the eprouvette.

162. *November* 18, 1789. *The same continued.*

Barometer 29·33 ; Thermometer 48.

N°	Dist. gun	Powder	Vibra. pend	Point struck	Plugs wt	Values of p	Values of g	Velocity of the ball	Veloc. reduced
	feet	oz	c	inches	oz	lbs	inches	feet	feet
1	30	4	82	92·7	6	1839	79·8		
2	•	•	77	92·6	4	42	8	669	670
3	•	•	78	90·8	4	45	8		
4	80	•	79	92·0	8	48	8		
5	•	•	71	85·8	5	51	8	670	669
6	•	•	74	85·0	4	53	8		
7	130	•	77	92·7	5	56	8		
8	•	•	83D	87·0	3	59	9		
9	•	•	83D	85·8	5	62	9	651	650
10	•	•	75	90·1	6	64	9		
11	•	•	67	82·6	5	67	9		
12	180	•	70	85·8	4	70	9		
13	•	•	71	88·5	7	73	9	641	641
14	•	•	67	83·8	5	75	9		
15	230	•	75	92·4	4	78	80·0		
16	•	•	75	92·2	4	81	0	653	652
17	•	•	76½	93·9	3	84	0		
18	30	12	150	94·4	6	87	0		
19	•	•	147	91·8	7	89	0	1296	1291
20	•	•	147	89·7	8	92	0		
21	80	•	133	88·1	9	95	0		
22	•	•	139	89·9	9	98	1	1247	1243
23	•	•	136	87·6	8	1900	1		
24	130	•	131	89·1	10	3	1		
25	•	•	137	95·9	10	6	1	1226	1222
26	•	•	130	85·9	8	1856	79·8		
27	180	•	132	86·7	9	59	9		
28	•	•	136	88·2	9	62	9	1214	1211
29	•	•	138	90·9	8	65	9		
30	230	•	132	88·3	9	68	9		
31	•	•	131	86·6	10	71	9	1205	1200
32	•	•	132	86·7	9	73	9		

Here the balls were 2·78 inches diameter; the first two weighed 2lb 15oz, the next 13 weighed each 2lb 14¾oz, and all after 2lb 14½oz. The three trials of the powder in

the eprouvette gave 35, $35\frac{1}{6}$, $35\frac{1}{3}$, the medium 35·1, each 2 oz; 4 oz gave $84\frac{3}{4}$. At the 25th shot a large iron band was broken and fell off from the lower part of the pendulum, weighing 52lb, which was deducted in the following values of *p* and *g*. In the last column the velocities are reduced to the state of the barometer 30, the ball 3lb, and the eprouvette 36.

163. *November* 19, 1789. *The same continued.*

Barometer 29·50; Thermometer 46.

N°	Dist gun	Powder	Vibr pend	Point struck	Plugs wt	Values of *p*	Values of *g*	Velocity of ball	Veloc reduced
	feet	oz	c	inches	oz	lb	inches	feet	feet
1	330	12	131	94·7	9	1876	80·0	1119	1105
2	·	·	125	91·4	9	79	0		
3	·	·	120	86·0	10	82	0		
4	430	·	84D	87·9	3	84	0	grazed	

After discharging many more balls from this last distance, without striking the pendulum, but always grazing or striking the screen of timber, the further attempts were declined. The balls used to-day weighed only 2lb 14oz. The powder by the eprouvette 35, $35\frac{1}{2}$, $35\frac{3}{4}$, the medium 35·4.

The same 3-pounder gun was fired, with 4 oz of powder, at 30 feet from the pendulum, with the ball laid, in the bore, at different distances from the muzzle, the powder being at the bottom, to find the loss of velocity. The weight of each ball 2lb 15oz, and the diameter 2·78 inches. The annexed tablet shows the chief circumstances; the velocities will be nearly in the ratio of the numbers in the 4th column divided by those

N°	Ball from muzzle	Point struck	Vibr pend	Plugs wt
	inches	inches	c	oz
1	$64\frac{1}{2}$	90	76	10
2	$59\frac{3}{4}$	90	69	7
3	$53\frac{3}{4}$	90	67	6
4	$47\frac{1}{4}$	90	62	5
5	$41\frac{3}{4}$	88·6	59	3
6	$35\frac{1}{4}$	90	55	3
7	$29\frac{3}{4}$	88·5	46	
8	$23\frac{3}{4}$	90	43	
9	$17\frac{1}{4}$	90	40	
10	$11\frac{3}{4}$	90	34	
11	$5\frac{3}{4}$	81	18	
12	0	76	13	

in the third, or indeed nearly as the former numbers only, as the latter are nearly equal. The 10th and 12th balls rebounded back again from the pendulum.

When the broken off iron band was restored to its place again, the pendulum oscillated $397\frac{1}{2}$ times in 10 minutes, which gives $n=39\frac{3}{4}$, nearly the same as at first.

164. *November* 20.

Weighed the pendulum, and balanced it, for the centre of gravity. Its weight was found 1976lb. The centre of gravity measured 80·3, which at the beginning was 79·3, the change being 1 inch, which is equally distributed among all the numbers, but reduced by $\frac{3}{10}$ after the iron band broke off. On the whole it appears that,

Wt of balls lodged	246lb
Plugs ditto . .	$34\frac{1}{2}$
Sum of both	$280\frac{1}{2}$
Wt at first	1748
Sum	2028
Wt at the end	1976
Lost by evaporation and splinters	52

The sum of all the points struck 6154, divided by the number of them 69, gives . . : . 89·2
Cent. of gravity at first . 79·3
the difference of these is 9·9

Then, as 1976 : 228 : : 9·9 : 1·1 the computed change in the centre of gravity, being very nearly the same as it measured.

A Synopsis of the two Sets of Experiments with the Long Three-pounder Gun this year.

165. Experimented Velocities with various Charges and Distances. The ball 3lb wt, and diam. 2·78 inches.

Barom. 30; *and Powder* 36.

Dists.	24 oz	16 oz	12 oz	8 oz	4 oz
30	1722	1568	1291	1061	670
80	1695	1473	1243	1048	669
130	1604	1471	1222	1015	650
180	1578	1410	1211	1015	641
230	1546	1384	1200	960	652
330	1498	1310	1105	922	
430	1300	1219		895	

Many of these numbers, it is evident, are irregular, which I suspect is chiefly owing to putting the powder into cartridges or bags. So that I think it would be advisable to try some experiments hereafter with the powder neatly put in loose, as I have found the benefit of that method in charging my new eprouvette, which acts much more regularly, since putting the powder in after that manner.

166. By omitting the three numbers on the last line, opposite to the distance 430, as the most doubtful of any, and making some small alterations in some of the numbers, at the same time interpolating another number in each column between the numbers for 230 and 330, the series of numbers will then run pretty regular, as will appear by their differences in small figures, in the following table.

TABLE OF REGULAR VELOCITIES.

Dist	24 oz		16 oz		12 oz		8 oz		4 oz	
30	1730		1555		1290		1060		680	
		48		41		31		24		11
80	1688		1514		1259		1036		669	
		46		40		30		24		10
130	1636		1474		1229		1012		659	
		44		38		29		23		10
180	1592		1436		1200		989		649	
		42		37		28		23		9
230	1550		1399		1172		966		640	
		40		35		27		22		
280	1510		1364		1145		944			
		39		34		26		22		
330	1471		1330		1119		922			

167. Taking then the mediums between every two of these velocities, for each charge of powder, place them in the first column of the annexed table; after which, in the 2d column, set the corresponding velocities lost in passing through 50 feet of air. Then, to these data applying the theorem $\frac{vv'}{32s}b$, where v denotes the middle velocities in the first column; v' the velocities lost, as in the 2d column; $s = 50$ feet, the common space; and $b = 3$lb $= 48$oz, the weight of the ball; with which data, the theorem $\frac{vv'}{32s}b$ becomes $\frac{48vv'}{1600} = \frac{3}{100}vv'$; which theorem will bring out the corresponding resistances in the 3d column; the first and second differences of which are set in the 4th and 5th columns, showing that the resistances are not far from being regular.

	Middle Velocities	Veloc lost	Resistances	First Diffs.	2d Diffs
	feet	feet	oz		
4lb 4oz	645	9	174	12	
	654	$9\frac{1}{2}$	186	13	1
	664	10	199	14	1
	675	$10\frac{1}{2}$	213		
8 oz	933	$21\frac{1}{2}$	604	28	
	955	22	632	29	1
	978	$22\frac{1}{2}$	661	30	1
	1001	23	691	31	1
	1024	$23\frac{1}{2}$	722	32	1
	1048	24	754		
12 oz	1132	26	883	56	
	1159	27	939	58	2
	1186	28	997	60	2
	1215	29	1057	63	3
	1244	30	1120	66	3
	1275	31	1186		
16 oz	1347	34	1374	81	
	1382	35·1	1455	89	8
	1418	36·3	1544	97	8
	1455	37·6	1641	107	10
	1494	39	1748	117	10
	1535	40·5	1865		
24 oz	1491	39	1744	120	
	1530	40·6	1864	130	10
	1571	42·3	1994	142	12
	1614	44·1	2136	154	12
	1659	46	2290	167	13
	1706	48	2457		

But now, to adapt the resistances to the series of velocities differing by the constant difference of 50, and to render them regular, it will be convenient to construct the curve of resistance, making the velocities the ordinates, and the resistances the corresponding abscisses; which gives the curve AH plate 2; from which we perceive that the numbers in the table above the velocity 900, are much too small, but that the rest of them run pretty regular; and when reduced to every 50 feet difference of velocity, came out nearly as in the following table.

Table of Resistances to a Ball of 3 lb weight, and 2·78 inches diameter. Barom. 30.

Velocity	Resist.	Diffs.	2 Diffs	Resists	Diffs	2 Diffs
feet	oz			lbs		
900	566	78		35·4	4·8	
950	644	85	7	40·2	5·3	5
1000	729	92	7	45·5	5·8	5
1050	821	98	6	51·3	6·2	4
1100	919	104	6	57·5	6·6	4
1150	1023	110	6	64·1	6·9	3
1200	1133	116	6	71·0	7·2	3
1250	1249	121	5	78·2	7·5	3
1300	1370	126	5	85·7	7·8	3
1350	1496	131	5	93·5	8·1	3
1400	1627	135	4	101·6	8·4	3
1450	1762	138	3	110·0	8·6	2
1500	1900	141	3	118·6	8·8	2
1550	2041	143	2	127·4	9·0	2
1600	2184	144	1	136·4	9·1	1
1650	2328	144	0	145·5	9·1	0
1700	2472			154·6		

General Observations on the Experiments of the year 1789.

168. The first set of these experiments was made to afford a comparison between guns of different lengths, the one short, and the other very long. They were both placed with the muzzle at 30 feet distance from the face of the pendulum, being charged each time with 16 ounces of powder, and the general weight of the balls 2lb 15oz. After a sufficient number of rounds with each gun, the medium velocities, computed from them, were as follow; viz, the long gun 1584 feet per second, and the short one 1371 feet. So that the velocity from the long gun exceeds that from the short one, by a quantity between the 6th and 7th part of the latter only.

The short gun was afterwards fired with the same charge of powder and balls, but blocked up behind the breech, so as to prevent entirely the recoil; and then the medium of the velocities was 1353 feet, being rather less than before, when its sliding carriage was permitted to recoil; but the difference is not more than may be supposed to happen between any one day's experiments and another; so that it may be safely inferred, that stopping the recoil makes no perceptible difference in the velocity of the ball discharged.

This comparison between the long and short gun being settled, the experiments were prosecuted with the long gun alone, that being sufficient for determining the other points of enquiry. In the course of a great many days' experiments, as before related, in which the gun was charged with various quantities of powder, and discharged at many different distances from the pendulum, the medium velocities, among three good ones of every sort, came out as below, in the following tables, where they are all reduced to the same circumstances of ball, air, and eprouvette, viz. an exact 3lb ball, the height of 30 inches of barometer, and the division of 36 on the eprouvette. The table of medium velocities in those circumstances, is as follows.

Tables of Velocities, arranged according to the distances.

Medium rnds	Dists of gun	Powder	Velocities
	feet	oz	feet
1	30	4	669
2	·	8	1061
3	·	12	1290
4	·	16	1568
5	·	24	1717
6	80	4	669
7	·	8	1048
8	·	12	1242
9	·	16	1473
10	·	24	1692
11	130	4	650
12	·	8	1015
13	·	12	1220
14	·	16	1471
15	·	24	1601
16	180	4	641
17	·	8	1015
18	·	12	1204
19	·	16	1410
20	·	24	1572
21	230	4	652
22	·	8	960
23	·	12	1197
24	·	16	1384
25	·	24	1540
26	330	8	922
27	·	12	1104
28	·	16	1310
29	·	24	1491
30	430	8	895
31	·	16	1219
32	·	24	1274

The table of the same, arranged according to the charges.

N°	Powder	Distances	Velocities
	oz	feet	feet
1	4	30	669
2	·	80	669
3	·	130	650
4	·	180	641
5	·	230	652
6	8	30	1061
7	·	80	1048
8	·	130	1015
9	·	180	1015
10	·	230	960
11	·	330	922
12	·	430	895
13	12	30	1290
14	·	80	1242
15	·	130	1220
16	·	180	1204
17	·	230	1197
18	·	330	1104
19	16	30	1568
20	·	80	1473
21	·	130	1471
22	·	180	1410
23	·	230	1384
24	·	330	1310
25	·	430	1219
26	24	30	1717
27	·	80	1692
28	·	130	1501
29	·	180	1572
30	·	230	1540
31	·	330	1491
32	·	430	1274

169. From this table it appears that the velocities continually decrease, as the distances increase; and though the distances are not sufficiently regular to admit of so accurate a determination of the air's resistance, as will yet lay a foun-

dation for a true system of practical gunnery, yet the following inferences may nevertheless be usefully deduced from them.

First. By comparing the first-mentioned velocities by the short 3-pounder, with the velocities of the long one in this table, loaded with one pound of powder, at different distances, it may be perceived that the superior velocity with the long gun, which was found to be near $\frac{1}{7}$ greater, is reduced to an equality with the short one, during the flight of the ball through only 230 feet, or less than 77 yards; and as the lengths of these guns are very nearly in the proportion of 2 to 1, which is as great a difference as ever occurs in service between any two guns of equal calibre, it fully accounts for the small advantage obtained in the ranges of shot, by any increased length which the limits of practice will admit of: and also how very subject to error any decision must be, in determining the velocities corresponding with a certain length of gun, which is founded on the extent of their respective ranges; since the irregularities in the shots' flight, added to the last-mentioned circumstance, must render them very uncertain criteria, in all cases where great velocities are concerned.

2dly. This table also affords a further confirmation of the small advantage, in point of range, which is obtained by increasing the charge, beyond what is necessary to communicate a certain velocity to the ball; since the increased resistance to great velocities operates so powerfully, that they are quickly reduced, and soon destroyed: for example, it appears by the table, that the velocity communicated by 16 ounces of powder, after the shot has passed through a space of 230 feet, is reduced to nearly the same with that of 12 ounces of powder, in a flight of 30 feet. It may also be observed, that the velocity with 24 oz of powder, at 180 feet distance, and that with 16 oz at 30 feet distance, are nearly equal; differing only by 4 feet per second, though at their first discharge from the piece they differ by as much as 149 feet per second.

3dly. From the foregoing table, it is evident also, that the velocities communicated by different quantities of powder, are nearly in the proportion of the square roots of those charges.—Also, by a due computation from the quantity of velocity lost in the several distances, the resistance of the air to the ball of 2·78 inches diameter, moving with several velocities, will be nearly as expressed in the foregoing table, in p. 126, where the first column shows the velocity per second, with which the ball moves, and the other columns show the corresponding resistances of the air, in ounces or pounds; that is, when a ball of that size moves with a velocity of suppose 1700 feet per second, it is resisted by the air with a force which is equal to the weight or pressure of 2472 ounces, or 154½ pounds; and so of the rest.

From this table of resistances it appears also, that there is a gradual and regular increase of resistance, as the velocity is increased, from the least to the greatest, and without showing the appearance of such a very sudden or abrupt change in the nature and quantity of that resistance, as Mr. Robins suspected might obtain. But that the law of resistance gradually and slowly increases like the velocity itself, probably on account of the increasing partial vacuum behind the ball in its flight, from the slowest motion, when the resistance changes as the square of the velocity nearly, up to about the velocity of 1200 or 1400 feet, when, the vacuum being completed, the law of increase appears to have attained its highest pitch, being then nearly as the $2\frac{1}{9}$ power of the velocity; after which it gradually decreases again more and more, as the velocity increases higher, till it arrive at about the $2\frac{1}{20}$ power, and perhaps still lower; which, among several others, is a law that was unknown till it was discovered by means of these experiments.

A few trials were also made with a 6-pounder gun, which shows the practicability of using still larger pieces of ordnance in such experiments. The dimensions of the gun were these: the length of the bore 57½ inches, and its diamter 3·67. The balls on an average weighed 5lb 13oz, and

measured $3\frac{1}{2}$ inches diameter. The gun was charged with 8, 16, and 24 ounces of powder, and was placed at 80 feet distance from the pendulum, which was struck by the balls with the following medium velocities, viz.

Powder, ounces . . .	8,	16,	24,
Velocities, feet . . .	703,	990,	1222.

From which, the same law of the charges and velocities, as formerly noticed, again takes place, viz, that the velocity is proportional to the square root of the charge of powder.

A great variety of experiments were also made with the 3-pounder gun, to obtain the extent of the ranges; but these turned out so exceedingly various and irregular, that no certain deductions can be drawn from them. The results and mediums of them, however, such as they were, are as below.

N°	Elevation	Powder	Time flight	Range	Angle of fall
	degr	oz	secs	yards	degr
1	20°	8	12	1600	
2	30	8	22	2560	
3	45	2	16	1290	
4	45	4	$22\frac{1}{2}$	1950	
5	45	8		2400	
6	45	16		2700	61°
7	45	24		2800	
8	75	8		1900	
9	75	16		2000	
10	75	24		2200	

PENDULUM EXPERIMENTS OF THE YEAR 1791.

170. *June* 27, 1791.

A new pendulum having been prepared, of a greater length than any of the former, to be used with 6-pounder guns, as before had been employed with the 3-pounders and 1-pounders, to be discharged at different distances, for the resistance of the air, and other purposes; attended this day, and saw it weighed, &c, when its weight was found to be 14 cwt 2qr 8lb, or 1632lb. But, some days afterwards, before the beginning of the experiments, it weighed only 1629½lb.

171. *June* 28, 1791.

Attended the balancing and measuring the pendulum, to find its centre of gravity, &c; when it was found, that the diameter of the axis was 2·4 inches, and below the centre or middle of the axis,

the centre of gravity on one side of the block was 76·0
and on the other side or face it was . . . . . 75·7
the mean between them therefore was . . . . 75·85

172. *June* 29, 1791.

Attended the suspending the pendulum, and dividing and marking its face into inches, for the easier measuring the distances of the points struck by the balls; also making it vibrate in small arcs, for the centre of oscillation, which it did for 10 minutes, during which time it made 401 oscillations, which therefore is at the rate of 40·1 per minute; so that the numeral value of the letter n is 40·1.

173. *July* 13, 1791.

Attended the suspending, vibrating, and measuring the gun, which was a long 6-pounder of brass, the bottom of its bore being hemispherical. It performed 38 oscillations in each minute.

Whole length of bore, to extremity of the bottom	80·5 inc.
Diameter of the bore	3·65
Centre of gravity below the axis of motion . .	85·8
Centre or axis of the bore below ditto	92·2
Weight of the gun alone	1370lb
Ditto with all the iron work	1618

174. *July* 14, 1791.

Barometer 30·0; Thermometer 66.

Began firing the above suspended gun, as follows.

N°	Powder	Vibration of gun	Vibration of pend	Point struck	Plugs wt	Values of p	Values of g	Values of n	Velocity of the ball
	oz			inches	oz	lb	inches		feet
1	32	649	458	84·7	18	1627	76·10	40·20	1686
2	·	630	452	87·4	11	1634	15	22	1620
3	·	635	465	88·2	10	1641	20	24	1658
4D	24	569	462	91·9	9	1648	25	26	1588
5	·	549	410	86·2	8	1654	30	28	1508
6	·	549	409	85·9	9	1661	35	30	1517

The gun's distance was 30 feet; diam. of balls 3·55 inches, wt. 6lb 1½oz. Length of the first charges was 9·8 inches; of the latter 8·3.

My Eprouvette.

Oz	Chords	Vers.	Arc.
2	42½	9	24°32′
2	42½	9	24 32

The powder of last year's experiments gave only between 35 and 36 recoil on the chords of the eprouvette, which answers only to 6 on the versed sines: so that the powder of this year is considerably stronger than before: yet the men, who had the care of it, asserted that it was the same that remained of the former; but I suspect there is some mistake on that point, and that it had been removed or changed, and that they either did not know it, or would not confess it.

The cartridges were now, and hereafter, made of fine lawn paper, nicely rounded at the bottom, so as to fit exactly the bottom of the bore, which is an exact hemisphere; and the balls were let close into the same paper bags, upon the powder, with a thread tied round the bag, both below the

ball and above it, to confine it in its place, near half sunk in the powder, and the top of the paper over the ball cut off, the neck at the tie being neatly folded down; all which circumstances render the effects of the firings much more uniform than heretofore.

The penetrations of the balls into the wood were so deep, that a sharp iron wire piercer could not be found long enough, to reach the balls in the pendulum block. After the first shot, the piercer or searcher was introduced the depth of 26 inches, without meeting the ball: so that the first penetration must have been more than 29½ inches.

After the experiments were ended this day, 2lb additional of wooden plugs were driven into the pendulum, to wedge it up close; which therefore must be added to its weight hereafter.

175. *July* 16, 1791.

Barometer 30; Thermometer 69.

The same experiments continued with the lower charges.

N°	Powder	Vibration of: gun	Vibration of: pend	Point struck	Plugs wt	Values of: p	Values of: g	Values of: n	Velocity of ball
	oz			inches	oz	lb	inches		feet
1	16	443	340	82·9	12	1668	76·40	40·33	1312
2	16	437	327	81·6	9	1676	45	35	1287
3	8	282	229	81·0	7	1681	50	37	911
4	8	279	220	80·5	9	1688	50	40	884

The balls and distance of the gun the same as before. The length of the 16 oz charge 6·75; of the 8 oz 5·3 inches.

My Eprouvette.

Oz	Chords	Vers.	
2	42½	9	These with the large grain used in the experiments.
2	42¼	9	
2	45	9½	This charge was with the fine grains

that were sifted out of the powder every day, with a peculiar sifting machine, the large grain being used in the experiments. Hence it appears that the fine-grained powder is stronger than the large-grained, at least in this quantity, 2 ounces.

On vibrating the pendulum after the experiments ended this day, for the centre of oscillation, it performed 405 vibrations in 10 minutes, or 40·5 per minute, which was 40·1 at the beginning. But by computing the variation in the value of n, from the change in the weight, it gives only a difference of 2: I have therefore assumed the value of n at first to be 40·2, and at the end 40·4.

Weighed and balanced the pendulum, when its weight and centre of gravity were found to be as below.

Centre of Gravity,	
On one side . .	76·65
On the other side .	76·80
The medium of both	76·72
Ditto at beginning	75·85
Medium nearly . .	76·3
Wt. added, changes the centre ·4, its half is	0·2
Correct at beginning	76·1
Ditto at the end .	76·5

For the Weight.		
Weight at first .		1629½
Ten balls lodged	60½	69
Plugs wt . .	6½	
Additional ditto	2	
The sum is .		1698½
Weight at the end		1694
Theref. lost by evaporation . .		4½

176. *July* 18, 1791.

Barometer 29·75; Thermometer 71.

Attended the weighing and balancing the gun, also fired the same charges of powder without balls, as had been used the last two days with balls.

No	Powder	Length Charge	Vibr. gun
	oz	inches	
1	32	6·8	294
2	32	6·5	295
3	24	5·3	220
4	24	5·3	221
5	16	3·7	141
6	16	3·7	140
7	8	2·9	63
8	8	2·8	64

My Eprouvette.

Oz	Chords	Vers.	
2	43	9½	large grains
2	43	9½	large grains
2	45	10½	small grains

The gun vibrated 38 + per min. See its weight, dimensions, &c, July 13.

177. *July* 21, 1791.

Barometer 29·89; Thermometer 67.

The pendulum having been repaired, and received a new core, I attended the weighing and suspending it. It weighed 1630lb, which being but 4lb less than before it was repaired, it may be presumed that its centres of gravity and oscillation are the same as ended with, on the 16th instant. Also suspended the medium 6-pounder gun, which was exactly the same diameter as the long one; and all its dimensions, weight, &c, were as below.

	lb
Weight of gun alone	1173
Ditto of iron work	248
Ditto both together	1421
	inches
Dist. centre of gravity	34·9
Dist. axis of gun	92·2
Length of bore	56·65
Diameter of bore	3·65
Vibration per minute	39—

Eprouvette.

Oz	Chord	Vers.
2	43	9¼
2	43	9¾

Discharged the following rounds.

Dist of the gun	30 feet.
Diam. of balls	3·55 inc.
Wt. of ditto	6lb. 1¼oz.

No	Powder	Vibration of gun	Vibration of pend	Point struck	Plugs wt	Values of p	Values of g	Values of n	Veloc. ball
	oz			inches	oz	lb	inches		feet
1	32	719	404	82·5	9	1690	76·50	40·4	1586
2	32	716	412	84·6	18	1697	54	4	1585
3	24	648	401	85·9	22	1704	58	4	1527
4	24		388	85·6	18	1712	62	4	1490
5	24	615	371	85·8	26	1719	66	4	1430
6	16	501	344	88·1	19	1726	70	4	1298
7	16	490	320	87·3	21	1733	74	4	1224
8	8	318	227	87·1	9	1740	78	4	874
9	8	323	232	88·6	15	1747	82	4	882

The Charges, oz.	32	24	16	8
Their Length, inc.	9·7	8·37	6·75	5·4

178. *July* 28, 1791.

Barometer 29·83; Thermometer 66.

N°	Powder	Length of charge	Vibration of gun
	oz	inches	
1	32	6·8	338
2	32	6·7	343
3	24	5·2	252
4	24	5·25	254
5	16	3·85	157
6	16	3·75	160
7	8	2·75	75
8	8	2·65	76

Fired the same medium gun with the same charges of powder as before, but without balls. The results are here annexed.

Eprouvette.

Oz	Chord	Vers.
2	43	$9\frac{1}{4}$
2	$43\frac{1}{4}$	$9\frac{1}{3}$

The pendulum having also been again repaired, with the addition of a new core, and strong iron bands across the back, to prevent the wood from bursting out behind; weighed it, and balanced for the centre of gravity, as below:

Its weight was .	1803lb
Cent. of grav. one side	76·7
Ditto the other side	77·2
Medium of both	76·95

And it may be presumed this last number is very near the truth, as the calculation on the additional weight of the iron cross bars above mentioned, alters the centre of gravity by ·4 of an inch, which added to 76·5, what it was before, makes it 76·9, almost the same as that annexed.

179. *Sept.* 14, 1791.

Barometer 30·2; Thermometer 70.

Suspended the long 6-pounder gun, and discharged it against the before-mentioned pendulum, from the distance of 200 feet, for the resistance of the air.

Eprouvette.

Oz	Chord	Vers.	Arc.
2	43	$9\frac{1}{4}$	24°50′
2	$43\frac{1}{3}$	$9\frac{1}{3}$	25 0
2	$43\frac{1}{3}$	$9\frac{1}{3}$	25 0

Distance of the gun, 200ft.
Diam. of the balls, 3·55 in.
Weight of ditto, 6lb $1\frac{1}{2}$oz.

No	Powder	Length charge	Vibr pend	Point struck	Plugs wt	Values of			Velocity of ball
						p	g	n	
	oz	inches	c	inches	oz	lb	inches		feet
1	32	9·9	371	82·5	20	1801	76·90	40·4	1560
2	32	9·9	359	81·8	17	8	93	·	1529D
3	32	9 9	389	84·4	19	16	97	·	1614
4	24	8·4	345	89·0	18	23	77·00	·	1362
5	24	8·4	327	86·9	22	30	04	·	1328D
6	24	8·3	324	80·6	20	32	07	·	1425
7	16	6·9	280	82·2	17	39	11	·	1238
8	16	6·8	298	90·6	15	46	14	·	1174
9	8	5·4	225	91·0	10	53	18	·	886
10	8	5·3	226	93·2	10	60	21	·	872

The 2d and 5th rounds are doubtful, owing to accidents: the ball dropped out at the 5th round. After this day's experiments, the pendulum was quite ruined.

It vibrated 40½ per minute.

	lb	oz
Its weight, July 28 was	1803	0
Weight of 9 balls .	54	13½
Ditto of plugs . . .	10	8
The sum is	1868	5½
But wt at last . . .	1866	0
Theref. lost only . .	2	5½

by evaporation; which is but a small quantity, considering how dry the weather had been: indeed it was all the time closely wrapped up in an oiled cloth, which probably in a great measure prevented the evaporation.

180. *Sept.* 21, 1791.

Attended the weighing and balancing the new pendulum, being an entire new block, of 52 inches long, and the face 23 inches by 17. Its weight was 1662lb. Dist. below bottom of axis to

Centre of gravity on one side . . .	74·8
Ditto on the other side	74·3
The mean theref. is	74·55
Add half diam. of axis	1·15
Centre of gravity below axis of motion	75·7

181. *Sept.* 22, 1791.

Barometer 30; Thermometer 60.

Fired the long 6-pounder, and also a light short one, against the above-mentioned pendulum, from the distance of 30 feet, with charges of 3lb and 2lb of powder; the length of the bore of this light gun being 57·6 inches. From this time, the guns were not suspended like a pendulum, but placed on a ship-gun carriage, with trucks running on a level platform, and the recoil of the gun and carriage noted down.

Eprouvette.

2 oz. $42\frac{1}{4}$ chord

2 $42\frac{3}{4}$

The pendulum oscillated 121 in 3 minutes. The first 4 shots were with the long gun; the rest with the light gun.

The ball's diam. 3·56; wt. 6lb $1\frac{1}{2}$oz.

No	Powder	Recoil gun	Vibr pend	Point struck	Plug's wt	Values of			Velocity of ball
						p	g	n	
	lb	inches	c	inches	oz	lb	inches		feet
1	3	67	482	87·0	20	1662	75·70	40·33	1749
2	3	69	479	83·2	15	1669	74	33	1826
3	2	48	431	86·1	16	1676	78	33	1596
4	2	49	456	87·5	16	1683	82	33	1669
5	3	83	445	82·8	63	1690	85	34	1728D
6	3	72	437	88·0	16	1700	89	34	1607
7	3	72	412	83·0	13	1707	93	34	1614
8	2	58	383	80·3	15	1714	97	35	1558
9	2	57	378	80·0	10	1721	76·00	35	1550

182. *Sept.* 24, 1791.

Barometer 29·98; Thermometer 60.

The following 4 rounds were fired against the pendulum, with the medium and the light 6-pounder guns, on a carriage, viz, the first two rounds with the medium gun, and the other two with the light one, both 5 feet guns, each round with $1\frac{1}{2}$lb of powder, and at 30 feet distance from the same pen-

dulum as left hanging Sept. 22; the balls as before, viz, diameter 3·56 inches, and weight 6lb 1½oz. The lengths of the bore and weight of the guns, were thus:

	Medium Gun.	*Light Gun.*	*Eprouvette.*	
			Oz	Chord.
Length of bore	56·65 inc.	57·6	2	42¾
Diam. of bore	3·65	3·65		
Wt. of gun	1173lb.	605	2	43

No	Length charge	Recoil gun	Vibr pend	Point struck	Plug's wt	Values of			Velocity of ball
						p	*g*	*n*	
	inches	inches	*c*	inches	oz	lb.	inches		feet
1	8 2	44	373	86·8	13	1728	76·04	40·35	1416
2	8·2	44	386	90·0	13	1735	08	36	1420
3	8·3	78	378	88·5	12	1744	12	36	1420
4	8 2	83	397	90·5	12	1749	16	36	1465

183. *Sept.* 26, 1791.

Took down the pendulum, to be repaired.

Its weight at first was . .	1662lb
Add balls and plugs, Sept. 22	66¼
Ditto Sept. 24	27¾
The sum of all	1756
It weighed now	1755
Theref. lost by evaporation only	1

184. *Sept.* 27, 1791.

The pendulum having been repaired, it was weighed and balanced this morning.

Its weight was . .	1759lb
Sheet-lead facing .	19
The sum of both	1778
Cent. of grav. below axis	76·3inc.
Ditto on the other side	76·1
Mean of both . .	76·2

This distance, 76·2, is exactly what it comes to, by calculation, viz, from the weight of balls and plugs lodged in the pendulum Sept. 22 and 24.

185. *Sept.* 30, 1791.

Barometer 30·15; Thermometer 53.

The powder, which had been hitherto employed, having been all expended, on trying several new sorts, for a fresh supply, by means of the eprouvette, none of them could be found so strong as the former was. But, by sifting the large grains out of one of the best of them, the remaining small grains of it came up nearly to an equal strength with that before employed; and with this new kind it was that the following experiments have been made, viz, with the long 6-pounder gun, at 115 feet distance from the pendulum, the gun being mounted on the truck carriage. The balls, as before, weighed 6lb $1\frac{1}{2}$oz, and diameter 3·55 inches.

Eprouvette.	
2oz.	42
2	$42\frac{3}{4}$

After the experiments the pendulum vibrated 80 in 2 min. or 40 per min. so that $n = 40$ in all the rounds.

No	Powder	Length charge	Recoil gun	Vibr pend	Point struck	Plug Wt	Values of p	Values of g	Velocity of ball
	oz	inches	inches	c	inches	oz	lb	inches	feet
1	48	12·9	56	396D	85·0	12	1778	76·2	1597D
2	48	12·9	67	444	86·2	12	1784	2	1771
3	48	13·0	65	448	90·3	12	1791	3	1715
4	32	9·9	47	382	82·0	11	1797	3	1616
5	32	9·9	49	410	87·2	12	1804	3	1637
6	24	8·5	33	356D	83·5	14	1810	4	1491D
7	24	8·4	31	322	80·5	11	1817	4	1405
8	24	8·3	32	340	83·6	11	1823	4	1433
9	16	6·9	22	276	79·9	8	1829	5	1223
10	16	6·9	20	304	89·5	11	1836	5	1207
11	8	5·4	9	214	88·4	15	1842	5	863
12	8	5·5	9	211	88·2	11	1849	6	857

After the experiments the pendulum weighed 1855lb

Now at beginning it weighed . . . 1778

Add for 12 balls . . . . . $78\frac{1}{8}$

Ditto for the plugs . . . . . $8\frac{3}{4}$

Sum of these 1860

Deduct for loss of lead facing . . . 5

Remains the same wt. as at first . . 1855

So that nothing has been lost by evaporation, owing probably to the pendulum having been kept closely covered up with the painted cloth.

186. *Oct.* 10, 1791.

Barometer 29·27; Thermometer 57.

The pendulum having been again repaired, and its face covered with a new sheet of lead, the following rounds were fired from the long 6-pounder, mounted as before, at 285 feet distance, with balls weighing each 6lb 1¾oz, and diam. 3·55. The pendulum weighed 1861lb, and the medium of its centre of gravity measured 77 inches, which I suspect is erroneous, as the pendulum is nearly the same weight as before, when that centre was only at 76·6; I have therefore assumed the medium 76·8 to begin with It vibrated 40 per minute.

Eprouvette.

2 oz 42¼ chord.

2 42¼

N°	Powder	Recoil gun	Vibr. pend	Point struck	Plugs wt	Values of			Velocity of the ball
						p	*g*	*n*	
	oz	inches	c	inches	oz	lb	inches		feet
1	48	49	374	85·8	16	1861	76·8	40	1571D
2	48	63	432	92·9	15	1867	8	·	1674D
3	48	54	416	91·5	14	1864	8	·	1641
4	32	36	329	79·5	13	1869	8	·	1496
5	32	36	358	84·6	12	1845	8	·	1513D
6	24		277	74·7		1852	7	·	1329D

N° 1 is doubtful.—At N° 2 the ball came out at the side of the pendulum, and struck an iron band, which it cracked, and then rebounded, also some splinters of wood flew out, both making the recoil too great.—At N° 5 some more splinters, and a piece of iron band burst off by the elasticity of the wood, weight 29lb, when the pendulum began its motion, which again made the vibration too great.—N° 6 also burst another band, and ruined the pendulum. So that there are only two or three indifferently good shots.

187. *Oct.* 22, 1791.

Another new pendulum block having been prepared, its length 5 feet, and face 24 inches by 18, had it weighed in the Foundery, and found it just 19 cwt, or 2128lb, including its facing of lead, which weighed 21lb, being much larger than any of the former pendulums.

188. *Oct.* 24, 1791.

Weighed the pendulum again, when its weight was found only 2121lb, having evaporated no less than 7lb in the two days it lay uncovered in the Foundery. Balanced it also for the centre of gravity, when it was found at both ends the same, viz, 77·7 inches below the middle of the axis of suspension.

189. *Oct.* 27, 1791.

Barometer 29·96; Thermometer 46.

Suspended the new pendulum, and fired the following rounds against it, from the same distance of 285 feet, with the same long 6-pounder gun, mounted as usual on a truck carriage, running on a platform.

Eprouvette.

2 oz.	$42\frac{1}{2}$	The constant value of g = 77·7
2	$42\frac{3}{4}$	Ditto of n = 39·8

No	Charge. wt	Charge. height	Recoil gun	Vibrat pend	Point struck	Plugs wt	Wt pend	Veloc of ball
	oz	inches	inches	c	inches	oz	lb	feet
1	32	9·9	22	299	88·1	14	2119	1412D
2	32	9·9	22				2117	
3	32	9·9	24	310	87·6	15	2115	1471
4	32	9·9	22	293	85·4	13	2117	1428
5	24	8·3	17	286	89·6	15	2119	1330
6	24	8·3	16	292	90·9	16	2121	1339
7	16	6·8	10	240	87·1	14	2123	1150
8	16	6·9	10	243	90·1	16	2125	1127
9	8	5·3	4				2127	
10	8	5·4	4				2120	
11	8	5·2	4				2113	
12	8	5·3					2106	

Before the pendulum was hung up this morning it weighed 2119lb, having lost only 2lb the last 3 days, owing to the dampness of the weather in that time. It oscillated 398 in 10 minutes, which gives $n = 39{\cdot}8$ per minute. The balls weight 6lb $1\frac{3}{4}$oz, and diameter 3·56 inches. The pendulum proved to be of bad unseasoned wood, so damp indeed that the water dropped out very fast, after the balls entered; and many splinters flew out at every shot. The first and second rounds are doubtful, as both the balls came out: the first penetrated 42 inches, before it came out. The next 6 balls lodged, but the last 4 all came out, by striking near the edges, and ruined the pendulum by breaking several of the bars.—After the experiments were ended, the pendulum weighed 2099lb.

Now at first it weighed	2119lb.
Add for 6 balls lodged	$36\frac{3}{4}$
Ditto for 7 plugs	$7\frac{1}{4}$
The sum is	2163
Being less than at first	64

which must have been lost in splinters, bars, and lead facing, chiefly in the last 4 rounds.

It may be observed that the recoils are nearly, but a little more than, in proportion to the charges of powder.

190. In the following table are exhibited the mediums of the several circumstances and computed velocities, as brought together, and arranged in the order of the dates, after having, with great labour and patience prosecuted these experiments as far as practicable, with respect to the magnitude of the pendulums, the size of the balls, and the distance from the gun.

Synopsis of the foregoing Experiments brought together.

Date	Gun	Barometer	Powder	Eprouvette	Dist gun	Vibration gun with ball	Vibration gun without bll	Veloc. comput'd	Velocity reduced
July			oz		feet			feet	feet
14	Long gun	30·0	32	42·5	30	638	294	1655	1663
14		·	24	·	·	556	220	1513	1521
16		·	16	·	·	440	140	1300	1306
16		·	8	·	·	280	63	898	902
21	Medium gun	29·9	32	43	30	718	341	1585	1585
·		·	24	·	·	632	253	1460	1460
·		·	16	·	·	496	159	1261	1261
·		·	8	·	·	321	76	878	878
Sept. 14	Long gun	30·2	32	43·2	200	Recoil or slide.		1567	1564
·		·	24	·	·			1399	1396
·		·	16	·	·			1206	1204
·		·	8	·	·			879	877
22	Long light	30·0	48	42·6	30	68		1788	1792
·		·	32	·	·	49		1633	1636
·		·	48	·	·	72		1621	1624
·		·	32	·	·	57		1554	1558
24	Med. light	30·0	24	42·9	30	44		1418	1419
·		·	24	·	·	81		1442	1443
30	Long gun	30·2	48	42·4	115	66		1743	1755
·		·	32	·	·	48		1627	1639
·		·	24	·	·	32		1443	1454
·		·	16	·	·	21		1215	1224
·		·	8	·	·	9		860	867
Oct. 11	Long gun	29·3	48	42·3	285	55		1629	1644
·		·	32	·	·	36		1505	1522
·		·	24	·	·	36		1329	1342
27	Long gun	30·0	32	42·6	285	23		1450	1454
·			24	·	·	17		1335	1339
·			16	·	·	10		1139	1142
·			8	·	·	4			

In this table, the numbers in the column of Eprouvette, denote the chord of vibration with 2 oz of powder, the radius being 100. The vibrations of the guns, are chords to radius 1000. The balls were all 3·55 or 3·56 inches dia-

meter, and 6lb 1½oz weight. In the last column, the velocities of the balls are all reduced to the state of the barometer 30, and Eprouvette 43.

191. The series of velocities, being selected from the preceding synopsis, for each gun and distance, will stand as in the following tables, as adapted to the ball of 3·55 or 3·56 inches diameter, 6lb 1½oz weight, with the strength of the powder 43 by the Eprouvette, and the Barometer 30 inches.

The light gun.			The medium gun.		
Distance	Powder	Velocity.	Distance	Powder	Velocity.
feet	oz	feet	feet	oz	feet
30	48	1624	30	32	1585
·	32	1558	·	24	1460
·	24	1440	·	16	1260
				8	877

The Long Gun Mediums, and reduced Regular.

Dist.	3lb Powder		2lb Powder.		1½lb powder		1lb powder	
feet	Comp. veloc.	Regul veloc.	Comp. veloc.	Regul. veloc.	Comp. veloc.	Regul. veloc.	Comp. veloc.	Regul. veloc.
30	1792	1810	1650	1673	1521	1506	1306	1306
115	1755	1745	1639	1616	1454	1453	1224	1259
200		1685	1564	1561	1396	1402	1204	1214
285	1644	1627	1522	1508	1342	1353	1142	1171

192. As the series of decreasing velocities are not quite regular, the numbers are altered here and there by certain very small quantities, which, without materially affecting the whole velocity itself, has the effect of making the series become regular, and so the more easily leading to the regular law of resistance. These corrected, or regulated velocities, are set in columns, each immediately after its corresponding column of irregular or calculated velocity; and those regular columns being extracted from the rest, and brought together by themselves, with their correspondent distances and differences, will stand as follows.

Regular Velocities of the Long Gun, and their Differences.

Dists.	Diffs.	3lb powder.		2lb powder.		1½lb powder.		1lb powder.	
		Veloc.	Diffs	Veloc.	Diffs.	Veloc.	Diffs.	Veloc.	Diffs.
30		1813		1676		1506		1306	
	85		65		58		52		47
115		1748		1618		1454		1259	
	85		62		56		50		45
200		1686		1562		1404		1214	
	85		59		54		48		43
285		1627		1508		1356		1171	

193. Then, by applying to these numbers the usual theorem, $r = \frac{vv'}{32s}b$, where r denotes the resistance to the middle velocity v, and v' the velocity lost in the space s, which here is 85 feet, with the mean velocity v, and b the weight of the ball, which in this case is 6lb 1¼oz, and diameter 3·55 inches; the resistances in lbs will be as in the 2d column of table 1 here below, to the mean velocities exhibited in the first column.

Table I.		Table II. Regular, in lbs and ounces.						
Mean veloc.	Comp. resist.	Regular veloc.	In ounces	Diffs.	2 diffs	Regular resist.	Diffs.	2 diffs
feet	lbs	feet	oz			lbs		
1193	115	1200	1840			115·0		
				190			11·9	
1237	125	1250	2030		9	126·9		·5
				199			12·4	
1283	135	1300	2229		8	139·3		6
				207			13·0	
1378	154	1350	2436		7	152·3		4
				214			13·4	
1428	160	1400	2650		6	165·7		3
				220			13·7	
1480	172	1450	2870		4	179·4		3
				224			14·0	
1535	185	1500	3094		3	193·4		2
				227			14·2	
1590	199	1550	3321		2	207·6		1
				229			14·3	
1647	214	1600	3550		1	221·9		1
				230			14·4	
1654	219	1650	3780		0	236·3		0
				230			14·4	
1717	239	1700	4010		-1	250·7		-1
				229			14·3	
1781	259	1750	4239		-3	265·0		-2
				226			14·1	
		1800	4465			279·1		

But as the velocities, in this table 1, are at unequal intervals, and the correspondent computed resistances contain some few very small irregularities, the numbers are reduced, in table 2, to equal intervals of velocity, differing by equal

differences of 50 feet, and the column of corresponding resistances are reduced by a due proportion, to uniformity and regularity, which appears by the differences in the last column of that table. That is, the numbers in the 2d column of table 2, show the true resistances sustained by a ball of 3·55 inches diameter, when moving through the air with the velocities in the preceding column, viz, a resistance of 1840 oz, or 115lb, when moving with a velocity of near 1200 feet per second, &c. &c.

The curve constructed from these velocities and resistances, is the curve AI, plate 2, being similar to those before constructed.

Observations on the preceding Experiments, in the year 1791, *with the Ballistic Pendulum.*

194. The experiments of this year were performed with three 6-pounder guns, all of equal diameter in the bore, but of different lengths and weights.

The few 6-pound balls that were fired against the ballistic pendulum, towards the conclusion of the last year's experiments, gave reason to hope that the pendulum might bear that weight of balls; and, by enlarging the apparatus, that it might even bear the shock of balls of a still larger size; the experiments of this year have therefore been prosecuted with balls rather higher than those of 6lb weight.

For this purpose, some pendulum blocks of wood were prepared, of a larger dimension than formerly, that they might not be perforated quite through by these heavy balls; with a proportionably strong apparatus of iron to bind them together, and to suspend them; the whole amounting to almost a ton weight.

The three 6-pounders with which these experiments were performed, were two short guns and a long one, the former of equal lengths, but differing in their weights, as follows.

Gun	Weight	Bore's	
		length	diameter
	lbs	inches	inches
Long	1370	80·5	3·65
Medium	1173	56·65	3·65
Light	633	56.6	3·65

These weights were those of the guns alone, without the iron apparatus for suspending them, when they were used in that way, which was the same for all the guns, and weighed 248lb.—The bore of the guns terminated at the bottom in the form of a hemisphere: and the charge of powder was put into very thin paper bags, shaped round at the bottom, so as to fit very nicely the end of the bore. The guns were sometimes suspended like pendulums, vibrating very freely, and at other times placed upon sliding carriages, to check the gun's recoil, thus reducing it to a very small quantity, to try if it would make any difference in the velocity with which the balls might be projected.—The weight of the balls was usually 6lb 1¼oz, and their diameter 3·55.

The states of the barometer and thermometer were taken, to judge of the effects of the weather on the powder, and on the resistance of the air against the balls.—The powder was proved every day by my new eprouvette, which shows very minute differences in the strength of the powder, and enabled us to regulate its strength, by keeping it always to the same degree. And indeed it proved very fortunate that we had such an eprouvette to employ, as it served to detect a change that had been made in the powder, which we should not otherwise have been aware of. For, the powder that had been used the year before gave only from 35 to 36 vibration of the eprouvette; whereas that which we began with this year gave from 42 to 43, constantly; and we expected that all the remaining barrels were of the same kind; but this did not turn out to be the case; for, after some day's experiments, on entering upon another barrel, we were surprised to find that the eprouvette showed it to be

of much less strength; and on trying the rest of the barrels, they were found to be all the same with this second one: so that the first barrel of this year, had been a single one, of a greater strength, that had accidentally been placed with the rest. As we had made a considerable progress in the experiments, it was material to find a powder of precisely the same strength with that we had last been using. But, as this was not to be found among these barrels, in their present state, we had recourse to the expedient of sifting some of the large grains out of the general mass, and, repeating the operation, we found the remainders gradually increase in strength, till at length the eprouvette showed the small grains to have the same strength, viz, between 42 and 43, as the former kind; a practice which we continued ever after. And thus the eprouvette not only detected a mistake, which would otherwise have spoiled the whole year's experiments, but also enabled us to discover a method of preparing a kind of artificial powder, to match the former.

195. The chief object of the experiments was, again further to ascertain the resistance of the air, by discharging these larger balls, as formerly the smaller ones, against the pendulum from different distances, and observing how much velocity was lost by flying through several spaces of air. Though several other objects were also proposed, and attended to, in the progress of the experiments; such as, the effects of either suspending the guns like a pendulum, or stopping their recoil; also the effect of different weights and lengths of guns; and the effect of the different charges of powder. The penetrations of the balls, in the blocks of elm wood, it was also proposed to attend to: but these turned out so deep, that a piercer could not be found long enough to reach the ball within the block, though it was sometimes introduced as much as 26 inches; so that the ball must have penetrated upwards of 29½ inches, in the solid wood, as the ball's diameter was more than 3 inches and a half.

In the tables at art. 191, 192, 193, are given general abstracts of the results of the experiments, being the mediums of the best rounds with the different charges, distances, &c, as computed from the experiments, and all reduced to the same circumstances, viz,

Ball's weight - - -	6lb $1\frac{1}{2}$oz,
Ditto diameter - -	3·55 inches
Eprouvette - - -	43
Barometer - - -	30

By comparing together the effects of the medium and light guns, of equal length nearly, it appears that there is no sensible difference between their velocities with equal charges of powder, though the one gun weighs nearly double of the other: whence it may be inferred, that their difference in weight has no sensible effect on the ball's velocity.—And the same thing happens by comparing together the velocities, when the gun is suspended like a pendulum, vibrating freely when fired, and when it is mounted on a sliding carriage which can recoil but very little, or even when the recoil is wholly prevented.—It also appears that the velocities arising from the firing of different charges, are nearly in the proportion of the square roots of the quantities or weights of powder.—By comparing together the long gun, of 80 inches bore, and the short ones, of 56, it appears that there is but little gained in velocity, in comparison of the length; the gain in velocity by the long gun being only from $\frac{1}{10}$ to $\frac{1}{15}$ of the whole.

The comparison between the long and short guns being settled, the experiments were prosecuted with the long gun only, as sufficient for determining the other points of enquiry.

196. It appears, from the table of velocities, with which the balls strike the pendulum from different distances, that they continually decrease as those distances increase; their differences showing the quantity of velocity lost by passing through certain spaces of air; from which losses of velocity

the quantity of the air's resistance is computed, by a theorem adapted to that purpose.

By comparing the velocities of the long gun and one of the short ones, with equal charges of powder, but at different distances, it may be observed, that the greater velocity of the long gun is very soon reduced, by the air's resistance, to an equality with that of the short one at its least distance; and that this equality takes place the sooner as the charge of powder is smaller. Thus, in the greatest or 3lb charge, the long gun's velocity is reduced to that of the short one, when it has passed through only 95 yards; but in the 2lb charge, in the space of 67 yards; the 1½lb charge in the space of 50 yards; and the 1lb charge in the space of about 38 yards: which fully accounts for the small advantage in the ranges of shot by a considerable increase in the length of the piece; and also how very subject to error any decision must be, in determining the velocities corresponding with certain lengths of piece, which is founded on their respective ranges; since the irregularities in the shot's flight, added to the above-mentioned circumstance, must render them very uncertain criteria, in all cases where high velocities are concerned.

The tables also furnish a further confirmation of the small advantage, in point of range, that is obtained, by increasing the charge beyond what is necessary to communicate a certain velocity to the ball; since the increased resistance to high velocities, operates so powerfully, that it is very soon destroyed, or greatly reduced.

Further, by a proper computation from the quantity of velocity lost in the several distances, the resistance of the air to a ball of 3·55 inches diameter, moving with several velocities, will be as shown in the last table in art. 193, where it may be observed that the columns of resistances have properties similar to those that have been before remarked on the experiments with the smaller balls; which is further remarkable, as, upon trial it is found, that the resistances against the different balls, with equal velocities, are very

nearly in the proportion of their surfaces, or, which is the same thing, as the squares of their diameters, being but very little higher, as might be expected. And this circumstance is a proof both of the accuracy or regularity of the experiments themselves, and also of the theory of the resistances of the medium, in this instance.

TRACT XXXV.

ON A NEW GUNPOWDER EPROUVETTE.

Art. 1. In the latter part of the preceding tract, it has appeared that the eprouvette, which had been used, proved of very essential service to the success of the experiments there described; so much so indeed, that, but for the use of it, the object of the experiments would have been in a great measure defeated, as no other known eprouvette could have been safely substituted for it, or have sufficiently answered the purpose in hand. It may be important here to give a description of the structure and use of this eprouvette, which may be of great and general benefit, wherever the quality of gunpowder is to be ascertained, as I know of no other machine by which that object can be effected, with so much ease and certainty, and dispatch; the facility and quickness of execution being such, that the weighing the parcels of the powder is the chief part of the operation; and the accuracy and uniformity are such, that a number of successive trials do not commonly differ by more than a small fraction of a degree, out of an extent of 40 or 50 degrees on the instrument.

2. The principle of this machine is remarkably simple; being nothing more than a small gun or mortar, suspended on an axis, which being charged with a small quantity of powder, without ball, and fired, the degrees of its vibration or recoil are easily measured, which give the quality or strength of the powder. The idea of this simple and accurate eprouvette was first started by the celebrated Mr. Robins, in his tracts on Gunnery, where he justly mentions it in very high terms. An eprouvette was afterwards, I believe, made on this principle, and probably from this notice, by the French chemist M. Baumè. It was described and represented, it seems, in his Chemistry, but in a form so very incommodious, as to render the beautiful simplicity of its principle of little use. My construction is totally different, is extremely easy and convenient, and has nothing in common with the former, but Mr. Robins's principle of an oscillating recoil.

3. The idea of this machine was suggested to Mr. Robins by the various experiments he made on the force of fired gunpowder, especially with his ballistic pendulum; and the occasion of my own machine being made, was as follows. Having had the honour to accompany Major (now General) Congreve, on several trials of experiments, to ascertain the strength of gunpowder, by means of different eprouvettes; who being commonly unsatisfied with the methods and instruments employed, and being always anxious to make improvements in every thing relating to the line of his profession and duty, he was pleased to urge me, by repeated requests, to turn my thoughts to the subject of the eprouvettes, and to endeavour to improve them, or to invent one, that might be at once, if possible, correct, as well as easy and expeditious in its operation. In one of our conversations on this subject, he showed me the drawing of an eprouvette, which he informed me was described and represented in M. Baumè's Chemistry, for the purpose of proving gunpowder. I immediately recognized this machine as acting on the same principle as that of Mr. Robins, which had

been published more than 40 years before; but I soon perceived also, that Baumè's machine was very ill adapted, both for convenience, and for accuracy of results.

4. In consequence of such communications, after various projects and essays, I at length succeeded to my utmost wishes, in accomplishing the project of a permanent eprouvette, by means of which, gunpowder may be proved, not only in an easy, commodious, and expeditious manner, but also with accuracy and uniformity of results; as has been verified on many occasions, by very numerous trials, and by the general approbation of all the most competent and disinterested judges, who have been witnesses of the experiments made with it. This machine may be described as consisting of either a small brass mortar, or gun, suspended by a metallic stem or rod, turning by an axis on a firm and strong frame, by means of which the piece oscillates in a circular arch. A little below the axis, the stem divides into two branches, reaching down to the gun or mortar, to which the lower ends of the branches are firmly fixed, the one near the muzzle, and the other near the breech of the piece. The upper end of the stem is firmly attached to the axis, which turns very freely by its extremities, in the sockets of the supporting frame; by which means the gun and stem vibrate together in a vertical plane, with a very small degree of friction. The piece is charged with a small, but proper quantity, of the powder to be proved, without any ball, and then fired; by the force of which the piece is made to recoil or vibrate, describing an arch or angle, which will be greater or less, according to the quantity or strength of the powder.

5. To measure the quantity of this recoil or vibration, and consequently the strength of the powder, a circular brazen or rather silvered arch, of a convenient extent, and of a radius equal to its distance below the axis, is fixed against the descending two branches of the stem, and graduated into divisions, according to the purpose required to be answered by the machine, viz, into equal parts, if we would know only

the angle of vibration, as measured by the simple equal degrees of a circle; or into unequal parts according to the chords, or to the versed sines of the arcs, to measure either the velocity of the vibration, or the force and strength of the powder: the arch in my instrument had all those three scales of divisions on it. The divisions in these scales, answering to the angle of any recoil, are pointed out by a concentric index, fitted on the axis of vibration, by means of a round hole or socket, with which it embraces pretty closely the round part of the axis of the stem, but capable of being turned easily about it by the hand. By means of a spring, the round end or socket of this instrument is pressed sideways, along the direction of the axis, always moderately tight against the socket of the stem, which is firmly brazed to the same axis; thus connecting the index and the stem slightly together; by which means, these two always vibrate in conjunction with the arch, unless when the index is stopped by some obstacle. When the machine is at rest, and the index brought to point to the beginning of the divisions on the arch, an additional piece fixed on the index bears against a stop-bar, fixed across the frame of the machine; so that, when the powder is fired, the gun and arch together vibrate backwards, leaving the index at rest, bearing still against the stop, and the divisions of the arch passing by it, till the gun has recoiled to the utmost extent that the force of the explosion can impel it: then, returning again, it brings the index along with it (because of their friction in consequence of the pressure of the spring) still pointing to the proper recoil division on the arch, showing the extent of the vibration; which, on gently stopping the vibrations, is easily read off, and noted down.

6. The circumstances which are peculiar to this eprouvette, and by which it differs from all others, as far as known to me, are as follow.

1st. The convenient manner of placing the arch, which measures the recoil, below the axis of the machine.

2nd. The divisions on this arch being made, not only according to equal degrees, but also according to the chords

and versed sines of the recoil, by which the true proportions of the velocities of balls, or the strength of powder, is shown.

3rd. The manner of applying the index, making it bear with a gentle pressure against the side of the socket of the stem, by means of a spring, and fixing it by a stop, while the gun and arch make the first or greatest vibration backwards.

7. In the course of the year 1783, I procured to be executed in London, by skilful artists, under my direction, two of these eprouvettes. The first was a small model in brass, with a small gun of about 6 inches in length; which, notwithstanding the diminutive size, performed very satisfactorily, being charged with proportionably small quantities of powder. The other was a machine executed in wood, and of the full intended size, the gun nearly $2\frac{1}{2}$ feet in length; being intended to serve as a pattern, by which to construct the large machine in metal. Both these machines were placed in the grand Artillery Repository, under the care of Major Congreve, in Woolwich Warren, now called the Royal Arsenal. From this wooden pattern, the complete machine was soon afterwards constructed in metal, in Major Congreve's office, at my request, by order of the Duke of Richmond, then Master General of the Ordnance; who was also pleased to give orders that it should undergo proper trials, and the results to be reported to him; which was done accordingly, with very general satisfaction and approbation, by the board of field-officers. This machine, after many useful experiments had been made with it, was also placed in the Repository, beside the former models, to be in readiness for use on any occasion that might occur; but where they were mostly destroyed, together with most of the other contents, when that noble institution was unfortunately consumed by fire in the year 1802. I trust however that a new one may have been since constructed, and placed in the New Repository in the Barrack Field, as a substitute for the former, for the good of the service.

8. *Construction and Description of the Vibrating Eprouvette.*

Plate 3 shows the machine, and its several parts, in various positions, as fitted up with a brass gun, the length of which was 27½ inches, the diameter of its bore being 1·7 inc. and consequently capable of receiving an iron ball of near 11½ ounces in weight: the dimensions of all the other parts may be measured by means of the annexed scale of feet and inches. In the plate, fig. 1 exhibits a side view or elevation of the machine at rest, with its stand or support. Fig. 2 is a front view of the same; fig. 3 a perspective view, with the gun having vibrated backwards; fig. 4 is an orthographic projection of the frame or stand, as seen by an eye placed vertically over it; and fig. 5 shows some of the parts separately.

Fig. 1 then exhibits the manner of suspending the gun by its axis; of connecting it to the two steel branches *a*, *b*, of the stem; as also of the quadrant *cd*, to the same, with its index *feg*; where *eg* is the part of it which rests upon a stop or small bar (*kl* in fig. 4) fixed across the upper part of the stand.

In fig. 2 are seen some of the same parts, with the steel axis *hi*, upon which the gun and arch vibrate.

In fig. 3 are seen the same things, each denoted by the same letters as in all the other figures.

In fig. 5 are exhibited the steel axis *hi*, the axis, *feg*, of the arch, with the part or arm *eg*, which rests on the stop part *kl*, to prevent the axis from recoiling back with the gun in the first vibration, and having two open notches near the end *f*, for viewing the different lines of divisions on the limb of the arch: *m*, *n*, show the socket and spring, for producing the gentle pressure of the socket *e* of the index, against the socket *o* of the axis, by means of the key *p* behind it.

9. Though the machine is here represented as fitted up with a small gun, it is to be understood that it may be equally well made with a small mortar, or a very short gun; which

will probably be more convenient in several respects, as the shortness will render it easier to load and clean, as well as more commodious in the act of vibrating; besides, the action of the fired gunpowder on it may be considered as only one single impulse, in which case the strength of the powder may be considered as proportional to the velocity given to the piece, and which therefore will be proportional to the chord of the angle of vibration, and may thus be measured and estimated by it; and hence this one line of divisions only, viz, the line of chords, is all that may be necessary to be drawn upon the limb of the arch.

10. *To construct the line of Chords on the Arch.*

The radius *ef*, of the arch to measure the vibration, it may be convenient to make of about 12 inches in length; and the extent of the arch itself *cd*, about a quadrant or quarter of the circle. From the centre *e*, with the radius of 12 inc. suppose, scratch a quadrantal arc, along the face *cd* of the instrument, upon which arc the lengths of all the chords are to be laid down, the divisions of which will not be at equal distances, or of equal lengths, after the manner of the equal degrees on arches, but will continually decrease from the beginning, where they are largest, all the way to the end, where they are smallest. Probably the best way of laying down the divisions, will be as follows:

Take in the compasses the whole extent of the quadrantal arch, from the beginning at *c*, to the end at *d*, having marked those two points, either by dots or short transverse lines. Lay that extent, which will be the chord of 90 degrees, upon a straight line, drawn on some plate of metal, or other hard and smooth plane surface. Divide this right line into as many equal parts, as the divisions of the intended chords on the arch are intended to be, as for example 100 parts, which will probably be a convenient number, or sufficiently numerous. Then transfer these divisions, from the beginning of the line, to the arch, always from the beginning at *c*, to the extent that each may reach to along the arch, observing that

every chord must commence at the point *c*; viz, first lay off every 10th division in the following manner: Take the extent from the beginning of the line to the 10th division, and transfer it to the arch, by placing one foot of the compass on the beginning at *c*, turn the other round to the arc that was described, with which make there a mark, for the 10th chord division. Next take with the compass the extent of the first 20 from the divided line, and transfer it to the arc from the beginning at *c*. In like manner transfer all the other extents of complete tens, viz, 30, 40, 50, 60, 70, 80, 90, each extent from the beginning of the arc and of the divided line. In the same way transfer any of the intermediate series from the divided line to the described arc, every extent always commencing at the beginning of the line and arc, as the series 5, 15, 25, 35, &c, to 95. After the same manner let all the other intervals, from the beginning of the line and arc, be transferred, as 2, 4, 6, 8, &c, to complete the division of the arc to every 2 on the chords; or the whole series 1, 2, 3, 4, &c, to complete it to every single unit of the chords, if the space will allow of such minute divisions: lastly, number the divisions at every 5, with the numbers, 5, 10, 15, 20, 25, &c, to 100, which will complete the arch to measure the velocity of the recoiling gun, and the strength of the fired powder.

11. *Remarks and Directions for using the Eprouvette.*

This eprouvette, it may be remarked, is not confined to any one particular size, or even shape, as it may be either a short cannon or a mortar, the latter probably the best; and these may be either small or large. The present instrument, with a small cannon, is to be considered only as a model of the cannon sort, to be used and charged with 2 or 3 ounces of powder, and with which a multitude of experiments have been made, with perfect satisfaction; the results, even from this size, being remarkably regular. If it be desired to try large charges, then a larger mortar or gun may be employed in the same way; but the present small one is quite sufficiently

accurate and regular. Though the experiments that have been tried with this eprouvette, have mostly been made with 2 ounces of powder, as a convenient medium charge; yet it is equally accurate with either more or less than that quantity; and it is remarkable that it shows the slightest variation that has been made, either in the strength or the quantity of the powder.

The instrument is so expeditious, simple, and easy in practice, that any common soldier, or other person, can learn the use of it completely in five minutes, employing it with safety and expedition, and setting down a register of his trials. So that, we have thus united in the same machine, the valuable qualities of accuracy, uniformity, ease, and expedition.

The machine can easily be made too by any artist, from the preceding description of its construction. Or it can be executed at home, and sent to distant places, namely, the small parts of it, and there fixed to any small mortar or swivel gun.

In the use of this instrument, for the constant regular practice of proving gunpowder, a certain mark or number, of the divisions on the arch line, may be fixed on, as a standard for the strength of the powder: which mark, or number of divisions, may be ascertained by charging it with any powder of approved strength or quality. And when once that mark or number is known, the machine will uniformly rise to the same place, at every trial, either accurately, or within a small fraction, with the same quantity and strength of powder. But should any other quality of powder be used, the machine will show the difference in its strength, with the utmost nicety, whether above or below the proof.

12. *To practise with this eprouvette;* charge it with two ounces, or any greater or less quantity of powder, put in loose, without using any bag or cartridge, and without any ball, in the following manner. Pour the powder carefully into the ladle; then, as the piece hangs at rest in the hori-

zontal position, introduce the ladle with the powder carefully into it, to the very bottom of the bore; as the ladle lies in this state, let an assistant push the gun gently forward a little, to raise the muzzle higher than the breech, and while he supports it in this position, turn the ladle of powder half round, to let the powder drop all out in the bottom of the bore; after which, draw out the ladle, and introduce the other or flat end of it, with which press the powder lightly all together in the bore, and then let the gun gently down again into the horizontal position.

13. After the gun has been thus charged with the powder, and before firing it, the index must be set exactly to the beginning of the divisions on the arch. This is done, by first setting it near the mark with the hand, and afterwards bringing it exactly to it, by means of a screw in the other end of the index, above the stand of the machine at *g*, in this manner: with one hand press the upper or screw-end of the index down upon the bar of the stand, and at the same time with the other hand turn the screw backward or forward, and you will perceive the lower end of the index at *f*, by that means, move slowly backward or forward, till it come exactly to the beginning of the divisions.

14. Before firing the powder, observe to have the socket of the index *e*, which fits upon the horizontal axis of the machine, always pressed to it with a moderate degree of friction; so that it may not sensibly obstruct the vibratory recoil of the gun, and yet that the index may keep any position you give it, without slipping from the arch or the axis, as the gun returns again from the first vibration.

15. After the piece has been charged with the powder, and the index set exactly to the beginning of the divisions on the arch, as before described, it is to be fired by means of a thread of quick-match in the following manner: take about 3 or 4 inches of fine quick-match, and introduce one end of it into the vent till it touch the powder in the bore, leaving the remainder to hang loose on the outside; then set fire to this with the port-fire, plucking this quickly away,

lest it should be struck by the stem that suspends the gun, which would somewhat check the recoil. After the gun returns from the recoil, stop the vibrations gently, without touching the index, and look what division on the arch the index points to, or stands at, which number will show the strength of the charge. For more satisfaction, the same fired charge may be repeated two or three times, and the medium of all the numbers taken for the strength of the charge.

TRACT XXXVI.

RESISTANCE OF THE AIR DETERMINED BY THE WHIRLING-MACHINE.

ART. 1. THE method of determining the resistance of the air, as detailed in Tract 34, by firing cannon balls into the ballistic pendulum, it was found by experience could only be practised with the higher velocities; since, with those that are below 300 in a second, the balls could hardly ever be made to lodge in the block, but rebounded back, and thus defeated the success of the experiment. To get the resistances then to the smaller velocities, I had recourse to another method, viz, to that of Mr. Robins's whirling-machine, one of which I was fortunate enough to meet with ready made, being probably the identical one with which he made his experiments, or at least it was made by the same workman, under the directions of Mr. Ellicot.

2. With this machine, Mr. Robins made many experiments before the members of the Royal Society, with the view chiefly of confirming these two propositions: " 1st.

That the resistance of the air to a 12lb iron ball, moving with a velocity of 25 feet in a second, is not less than half an ounce avoirdupois. 2dly. That the resistance of the air, within certain limits, is nearly in the duplicate proportion of the velocity of the resisted body." This machine is described in the first volume of his Tracts, page 202, &c; whence I shall here borrow the description, nearly in his own words.

3. The two propositions above-mentioned, Mr. R. there says, " are in themselves neither unknown nor doubtful; but yet, as they are the basis of some other assertions, which have been hitherto constantly contested and denied, I thought it requisite to evince their veracity by more unquestioned and simpler methods, than have been hitherto practised: for this purpose, I therefore caused a machine to be made of the form represented in plate 4, which was most excellently executed under the direction of Mr. Ellicot, and was completely fitted for the use intended, by many contrivances, some of them not shown in the drawing, nor necessary to be particularized in this place. For, it is sufficient, for the purpose of the following experiments, to describe its general fabric, by observing that BCDE is a brass barrel moveable on its axis, and so adjusted by means of friction wheels, which are not represented in the drawing, as to have no friction worth attending to. The frame, in which this barrel is fixed, is so placed, that its axis may be perpendicular to the horizon. The axis itself is continued above the upper plate of the frame, and has fastened on it a light hollow cone AFG: from the lower part of this cone there is extended a long arm of wood GH, which is very thin, and cut feather-edged, and at its extremity there is a contrivance for fixing on the body whose resistance is to be investigated, as here the globe P; and, to prevent the arm GH from swaying out of its horizontal position, by the weight of the annexed body P, there is a brace AH of fine wire fastened to the top of the cone, which supports the end of the arm.

"Round the barrel BCDE there is wound a fine silk line, the turns of which appear in the figure, and after this line has taken a sufficient number of turns, it is conducted horizontally to the pulley L, over which it is passed, and then a proper weight M is hung to its extremity. If this weight be left at liberty, it is obvious, that it will descend by its own gravity, and will by its descent turn round the barrel BCDE, together with the arm GH and the body P fastened to it. And while the resistance on the arm GH and the body P is less than the weight M, that weight will accelerate its motion, and thereby the motion of GH and P will increase, and consequently their resistance will increase, till at last this resistance and the weight M become nearly equal to each other. The motion with which M descends, and with which the body P revolves, will not sensibly differ from an equable one. Whence it is not difficult to conceive, that by proper observations made with this machine, the resistance of the body P may be determined.

4. "The most natural method of proceeding in this investigation, is as follows. Let the machine have first acquired its equable motion (which, as will hereafter appear, will be usually attained in 5 or 6 turns from the beginning) and then let it be observed, by counting a number of turns, what time is taken up by one revolution of the body P; then taking off the body P and the weight M, let it be examined, what smaller weight will make the arm GH revolve in the same time, as when P was fixed to it; this smaller weight being taken from M, the remainder is obviously equal in effort to the resistance of the revolving body P; and this remainder being reduced in the ratio of the length of the arm to the semi-diameter of the barrel, will then become equal to the absolute quantity of the resistance. And as the time of one revolution is known, and consequently the velocity of the revolving body; there is thereby discovered, the absolute quantity of the resistance to the given body P, moving with a given degree of celerity.

5. "And note, that to avoid all exceptions, I have gene-

rally chose, when the body P was removed, to fix in its stead a thin piece of lead of the same weight, placed horizontally; so that the weight, which was to turn round the arm GH without the body P, did also carry round this piece of lead. This I did, lest it should be objected that the body P retarded the weight M by its quantity of matter, as well as by its resistance. But mathematicians will easily allow, that there is no necessity for this precaution.

6. "The measures of the parts of this machine, were as follow:

	Inches.
"The diameter of the barrel BCDE, and of the silk string wound round it, was - - - - - - - - - -	2·06
"The length of the arm GH, measured from the axis to the surface of the globe P, was - - - - -	49·5
"The body P, the globe made use of, was of pasteboard, its surface very neatly coated with marble paper, and was not much distant from the size of a 12lb shot, being in diameter - - - - - - - - - - -	4·5
"So that the radius of the circle described by the centre of the globe, was - - - - - - - - - - -	51·75

7. "When this globe was fixed at the end of the arm, and a weight of half a pound was hung at the end of the string at M; it was examined, how soon the motion of the descending weight M, and of the revolving globe P, would become equable as to sense. And with this view 3 revolutions being suffered to elapse, it was found, that

The next 10 were performed in -	$27''\frac{3}{4}$
20 in less than - - -	55
30 in - - - - - - -	$82\frac{1}{2}$
So that the first 10 were performed in	$27\frac{3}{4}$
2nd 10 in - - - - - -	$27\frac{1}{4}$
3rd 10 in - - - - - -	$27\frac{1}{2}$

"These experiments sufficiently evince, that even with half a pound, the smallest weight hereafter used, the motion of the machine was sufficiently equable after the first 3 revolutions.

8. " And now, to prove the two forementioned propositions, the following experiments were made; the times marked down being observed by several stop-watches, which rarely differed half a second from each other.

" The forementioned globe being fixed at the end of the arm, there was hung on in the situation M a weight of $3\frac{1}{4}$lb, and 10 revolutions being suffered to elapse, the succeeding 20 were performed in $21''\frac{1}{2}$.

" Then the globe being taken off, and a thin plate of lead, equal to it in weight, placed in its room; it was found, that instead of $3\frac{1}{4}$lb, a weight of 1lb would make it revolve in less time than it did before, it performing 20 revolutions, after 10 were elapsed, in the space of 19 seconds.

" Hence then it follows, that from the $3\frac{1}{4}$lb first hung on, there is less than 1lb to be deducted for the resistance on the arm, and consequently the resistance on the globe itself is not less than the effort of $2\frac{1}{4}$lb in the situation M; and it appearing from the former measures, that the radius of the barrel is nearly $\frac{1}{50}$ part of the radius of the circle described by the centre of the globe; it follows, that the absolute resistance of the globe, when it revolves 20 times in $21''\frac{1}{2}$ (which, if computed by the measures given above, comes out a velocity of about $25\frac{1}{4}$ feet in a second) it follows, I say, that the resistance of the globe in this case is not less than the $\frac{1}{50}$ part of $2\frac{1}{4}$lb, or than the $\frac{1}{50}$ part of 36 ounces: and this being considerably more than half an ounce, and the globe being nearly the size of a 12-pounder; it irrefragably confirms our first proposition, 'That the resistance of the air to a 12lb iron bullet, moving with a velocity of 25 feet in a second, is not less than half an ounce avoirdupois.'

9. " The next experiments were made with a view of examining the 2nd proposition. And, for this purpose, there were successively hung on, in the situation M, weights in the proportion of the numbers, 1, 4, 9, 16; and letting 10 revolutions first elapse, the observations on those that follow, were as follow.

"With 1lb, the globe went 20 turns in $54''\frac{1}{2}$
that is, it went 10 turns in $27\frac{1}{4}$
With 2lb, it went . . 20 turns in $27\frac{1}{2}$
With $4\frac{1}{2}$lb, it went . . 30 turns in $27\frac{1}{2}$
With 8lb, it went . . 40 turns in $27\frac{1}{2}$

"So that it appears, that to the resistances of the proportions of the numbers 1, 4, 9, 16, there correspond velocities of the resisted body, in the proportion of the numbers 1, 2, 3, 4; which proves with great nicety, within the limits of these experiments, the 2nd proposition, 'That the resistance of the air, is nearly in the duplicate proportion of the velocity of the resisted body. That is, it is 4 times as much, when the resisted body moves with twice the velocity; 9 times as much, when it moves with 3 times the velocity, and so on'."

10. Having thus given Mr. Robins's description of the whirling-machine, and the experiments that he performed with it, which will serve sufficiently to bring us acquainted with the nature of it; proceed we now to take account of my own numerous and varied experiments made with the same, in the years 1786 and 1787.——The machine having been cleaned, and put in proper order, so as to cause it to act very perfectly; I procured to be made three hemispheres of pasteboard, of different sizes; both to try the resistance to bodies of different magnitudes, and the difference of resistance between the plane circle and the convex side of the hemisphere. The weight and dimensions of these three hemispheres were as below.

N°	Diam. inches	Area of circle, in		Weight.	
		sqr. inc.	sqr. feet	oz	dr
1	4·75	17·7	$\frac{1}{8}$	2	5
2	5·1	21·2	$\frac{5}{34}$	2	11
3	6·375	32·0	$\frac{2}{9}$	4	3

Having made also three small and thin bits of lead, shaped feather-edged, to fix on the arm in a horizontal position,

when each hemisphere was taken off, to find the resistance to the arm, and to obviate any objections, for the same reason as Mr. Robins had done before; the arm was fixed on, with the brass wire, strained from the top of the axis to the end of the arm, to sustain it in the horizontal position.

May 1, 1786.

11. Began to take the necessary dimensions of the parts of the machine. Measured the diameter of the axis two different ways, and each way several times over, taking a mean of them all. One way was by means of a sliding shot-gage, which measured diameters very nicely to hundredths of an inch, by taking the diameter between two parallel arms, made to slide along a stem, divided into inches and 10ths, and these into 10ths again, or the whole into 100ths of inches, by means of a nonius. In this way I took, several times over, and always the same, the diameter of the bare cylinder, without the silk thread wrapped, and then again the diameter including the silk thread on both sides, which is 2 diameters of the thread more than the bare cylinder; viz,

Diameter of the cylinder alone	2·06 inc.
Ditto and double diam. of the thread . .	2·11
Theref. diam. of cyl. and 1 diam. of thread	2·085
And radius to middle of the thread . .	1·042

The other way was, by making the machine turn round exactly a certain number of times, and measuring the space descended by a weight at the end of the thread, as it unwound from the cylinder; then the space so descended being divided by the number of turns, gave the length of one round or circumference of the cylinder and thread; and this again divided by 3·1416, gave the diameter of the cylinder and thread included. This operation being often repeated, and always nearly the same, the medium of all gave for the diam. of cyl. and thread 2·092, therefore the radius of the same is 1·046, which is only ·004 part of an inch more than that by the former method; and the medium of the two is 1·044.

May 2, 1786.

Barometer 30·05; Thermometer 51.

12. Measured from the axis of the cylinder to the centre of each hemisphere, when fitted on the arm, and found them as below, on the line of radii; thence are found the following lines of circumferences, &c.

	N° 1.	N° 2.	N° 3.
	inc.	inc.	inc.
Radii . .	52·42	52·74	53·34
Diams. . .	104·84	105·48	106·68
Circum. .	329·36	331·38	335·14
Ditto feet .	27·47	27·62	27·93
Ratio of radii	50·2	50·5	51·1

The last line being the ratio of each of those radii to that of the barrel and thread 1·044, or the former divided by the latter.

The hemisphere n° 3 was then fitted on, and actuated by a small weight of only 1¾oz, or 1oz 12dr, that is 1¾oz suspended at the end M of the thread; opposing first the round side to the air, and then the flat side; when the number of rounds, and corresponding times, were observed as follows, both now, and in all the other experiments, by stop-watches, and a peculiar clock made for the purpose.

Hemisphere n° 3, with Weight at M = 28 dr = 1 oz 12 dr.

Plane Side Foremost.			Convex Side Foremost.		
Turns	Time	Diffs.	Turns	Time	Diffs.
1	0′ 27″		3	0′ 59″	
2	· 47	20	6	1 37	38
3	1 6	19	9	2 12	35
4	· 24	18	12	2 47	35
5	· 42	18	15	3 22	35
6	2 0	18			
7	· 18	18	3	35	means
8	· 36	18	1	$11\frac{2}{3}$	
9	· 54	$18\frac{1}{2}$	$\frac{3}{35}$	1	
10	3 $12\frac{1}{2}$	18	2·349 ft.	1	
11	· $30\frac{1}{2}$	$17\frac{1}{2}$			
12	· 48	17			
13	4 5	$17\frac{1}{2}$			
14	· $22\frac{1}{2}$				
1	18	means			
$\frac{1}{15}$	1				
or 1·552 ft.	1				

Note, that the resistance of the arm is still wanted to be deducted from these resistances, and then to be reduced from the circumference of the barrel to the centre of the hemisphere.

In these tables, the first column contains the number of turns made by the arm; the 2nd the corresponding times elapsed from the beginning; and the 3rd shows the differences of those times, answering to the differences of the turns in the first column.

May. 3, 1786.

Barometer 29·63; Thermometer 51.

13. Fitted on the arm the thin bit of lead only n° 3, of the same weight with the largest hemisphere; and after some time spent in levelling the instrument, (which was always done) suspended different small weights at the end of the thread, to discover which weight would give to the arm the same velocities as it had on the day before, when the hemisphere was on. It was found that it required about 10 drams

just to overcome the friction and vis inertia, and give a small motion to the machine. With 12 drams the average of the turns was from $11\frac{1}{2}$ to 12″ each, being nearly the same velocity as the convex side of the hemisphere had with the weight of 12 drams. But with the weight of 1 oz 8 dr = 24 dr, the average turn of the arm this day was made in about 6″. Here then each turn is made in about 12″ with 12 dr weight, and in about 6″ with 24 dr; that is, in this case, the double weight carries it about in nearly half the time. And hence it may be inferred, that a weight of about 8 dr would carry it round once in 18″, being the time of revolving the day before, with the plane of the circle foremost.

Hence then 16 dr or 1 oz, which is the difference between 28 and 12, is the reistance of the air, measured only at the circumference of the barrel, against the convex side of the large hemisphere, moving with a velocity of 2·394 feet in a second; and 20 dr, or $1\frac{1}{4}$ oz, being the difference between 28 and 8 dr, is the relative resistance of the air against the plane side, moving with a velocity of 1·552 feet per second. And each of these divided by 51·1, will give the absolute resistance acting at the centre of the body.

May 4, 1786.

Barometer 29·47; Thermometer 51.

14. Tried several weights actuating the hemisphere, with the plane side foremost, to give it the same velocity as on the 2nd of May, with the convexity foremost, viz, the medium of 1 turn in $11\frac{2}{3}$ seconds. In each trial, the motion became uniform after the first two or three turns; after which, it was permitted to go 12 times round, and the time noted down at every round. The differences of these times commonly agreed with the mean time, to half a second or less. But the mean time itself was very accurately obtained, by dividing the whole observed time of the equable rounds, by 12, the number of them. Which operations, with the times and weights employed this day, are as follows.

Actuating weights		Time of 12 turns	Time of 1 turn
oz	dr		
2	8	158″	13″$\frac{1}{6}$
2	9$\frac{1}{2}$	146$\frac{1}{2}$	12 ^{1}
2	10$\frac{1}{2}$	142	11 $\frac{5}{6}$
2	11	141$\frac{1}{2}$	11 $\frac{3}{4}$
2	12	140	11 $\frac{2}{3}$

Hence then, with the velocity of 1 turn in 11$\frac{2}{3}$ seconds, the arm and plane side of the body are resisted with the force or weight of 2 oz 12 dr; but, on the 2d of May, the resistance to the arm and round side of the body was 1 oz 12 dr; and, on the 3d of May, the resistance to the arm alone was 12 dr; all with the same velocity nearly. Subtracting now 12 dr, the resistance on the arm, from each of the other two, there remain 2 oz for the resistance to the flat side, and 1 oz nearly for that on the convex side; the latter being nearly the half only of the former, and therefore, in this case, nearly agreeing with the theory.

Dividing now this 2 oz by 51·1, the ratio of the radii of the barrel and centre of the body, there results $\frac{20}{511} = \cdot 039$ oz, for the absolute resistance of the flat surface of a circle, moved perpendicularly through the air, with a velocity of 2·394 feet in a second, the area of the circle being 32 square inches, or $\frac{2}{9}$ of a square foot.

May 8, 1786.

Barometer 29·66; Thermometer 56.

15. Tried several different weights, to bring the large hemisphere flat side, also the arm without it, to revolve each time in 10 seconds. Then again, the same body, both sides foremost, and the arm alone, all in 8 seconds, each turn: The particulars are as follows.

For 10 seconds.					For 8 seconds.			
Weights	Plane side		Arm only		Weights	Plane side	Round side	Arm alone
	12 turns	1 turn	12 tns	1 trn				
oz dr								
3 0	140″	$10\frac{5}{6}$	″	″	4 10	8″	″	″
3 2	126	$10\frac{1}{2}$			6 0	7		
3 4	123	$10\frac{1}{4}$			8 0	6		
3 $5\frac{1}{2}$	120	10			2 2		10	
1 0			108	9	2 14		8	
0 14			132	11	1 3			8
0 15			120	10	1 8			6

In the latter of these two tables, the first column shows the actuating weights, and the other three columns the number of seconds, at a medium, in which 1 revolution was made, with each of the two sides of the hemisphere, and the lead alone.

Now, by deducting each weight without the hemisphere, from the corresponding weight with it, there remain the weights for its two sides, as below.

For 10 sec.		For 8 sec.	
3 $5\frac{1}{2}$	2 2	4 10	2 14
0 15	0 15	1 3	1 3
2 $6\frac{1}{2}$	1 3	3 7	1 11

Where the former is nearly double of the latter, in each case, agreeably to theory.

May 9, 1786.

Barometer 29·56; Thermometer 55.

16. Used the large hemisphere with the round side foremost. The weights and times of each revolution as below.

Weight	Time	Weight	Time	Weight	Time	Weight	Time
oz dr	sec.	oz dr	sec.	oz dr	sec.	oz dr	sec.
4 10	$5\frac{5}{6}$	6 0	$4\frac{11}{12}$	7 12	$4\frac{1}{6}$	12 7	$3\frac{5}{24}$
4 9	$5\frac{3}{4}$	6 5	$4\frac{3}{4}$	8 1	$4\frac{7}{12}$	12 10	$3\frac{1}{4}$
4 11	$5\frac{5}{6}$	7 0	$4\frac{1}{2}$	11 5	$3\frac{3}{8}$	13 0	$3\frac{1}{6}$
5 12	5	7 8	$4\frac{1}{4}$	12 0	$3\frac{7}{24}$		

May 10, 1786.

Barometer 29·60; Thermometer ——

17. The large hemisphere with the round side again. The motion was always uniform after the first or second turn. The number of turns always 15, and the medium of the last 12 uniform turns is set down here below opposite the corresponding weights, the same as was done every day, and the titles of the columns being the same as in the foregoing table.

Weight		Time	Weight		Time	Weight		Time	Weight		Time
oz	dr	sec	oz	dr	sec	oz	dr	sec	oz	dr	sec
8	5	$4\frac{1}{12}$	9	11	$3\frac{9}{12}$	13	12	$3\frac{1}{12}$	15	8	$2\frac{10}{12}$
8	8	$4\frac{1}{24}$	10	1	$3\frac{8}{12}$	14	0	3	16	0	$2\frac{9}{12}$
8	12	4	10	8	$3\frac{7}{12}$	14	5	3	16	8	$2\frac{9}{12}$
9	0	$3\frac{11}{12}$	11	0	$3\frac{6}{12}$	14	10	$2\frac{11}{12}$	17	0	$2\frac{9}{12}$
9	6	$3\frac{10}{12}$	13	8	$3\frac{1}{12}$	15	0	$2\frac{11}{12}$	17	8	$2\frac{8}{12}$

May 11, 1786.

Barometer 29·64; Thermometer 60.

18. The large hemisphere again with the round side: every thing else as on the last day.

Wt.	Time	Wt.	Time	Wt.	Time	Wt.	Time
oz	sec	oz	sec	oz	sec	oz	sec
18	2·67	28	2·12	38	1·79	58	1·46
19	2·58	29	2·10	40	1·75	60	1·42
20	2·50	30	2·	42	1·75	64	1·33
21	2·42	31	2·	44	1·67	68	1·33
22	2·42	32	2·	46	1·58	72	1·29
23	2·33	33	1·92	48	1·54	76	1·25
24	2·25	34	1·92	50	1·54	80	1·21
25	2·25	35	1·83	52	1·50	84	1·17
26	2·25	36	1·83	54	1·50	88	1·17
27	2·17	37	1·83	56	1·50	92	1·13

With the next weight, viz, 96 ounces, the silk thread broke twice: so that 96 oz, or 6lb, was just the limit of its strength. It was not woven, but only a twisted cord, framed like a rope, of 3 strands; and was of $\frac{1}{50}$ of an inch in diameter.

May 15, 1786.

Barometer 30·27 ; Therometer 62.

19. Procured a stronger silken cord, and applied it to the following experiments. By measuring the diameters, &c, as before, on the 1st of May, the dimensions came out as below.

Diameter of cylinder alone . .	2·06
Ditto and 1 diameter of thread .	2·086
Ditto by descending weight . .	2·088
Medium of both ways, is . . .	2·087
Theref. the radius is as before .	1·044 nearly.

Then continued the last experiments, by increasing the actuating weight, from where they left off the last day, with the convex side, as below.

Weight	Time	Weight	Time	Weight	Time	Weight	Time
oz	sec.	oz	sec.	oz	sec.	oz	sec.
96	1·12	108	1·04	116	1	124	1
100	1·08	112	1·04	120	1	128	0·96
104	1·08						

So that 120 oz may be presumed to give 1 turn in 1 sec.

The same continued still with the round side.

oz dr	sec.	oz dr	sec.	oz dr	sec.	oz dr	sec.
4 10	5·8	3 7	6·9	2 10	8·2	2 0	10·2
8	5·9	6	7·0	9	4	1 15	6
6	6·0	4	1	8	6	1 14½	8
4	1	2	3	7	8	1 14	11·1
2	2	0	5	6	9	1 13½	3
0	3	2 15	6	5	9·1	1 13	6
3 14	4	14	7	4	3	1 12½	9
12	6	13	8	3	5	1 12	12·2
10	7	12	9	2	7	1 11½	6
8	8	11	8·0	1	9	1 11	13·0

Note, That the machine made always 15 turns, and the time of each turn was carefully noted. Then the time of the 5th turn being taken from that of the 15th, left the time of 10 turns; and this by pointing off the first figure for a decimal, gave the time of 1 turn registered in the table as above,

20. *May* 16, 1786.

Barometer 30·14; Thermometer 61.

Began with the flat side of the large hemisphere, in order to go regularly through all the intervals of rotation, from 12 sec. to 1 sec.

Weight	Time	Weight	Time	Weight	Time	Weight	Time
oz dr	sec.	oz dr	sec.	oz dr	sec.	oz dr	sec.
2 10	13·1	2 12	12·1	3 0	11·4	3 6	10·5
10½	12·7	13	11·9	1	11·2	8	10·3
11	12·5	14	11·8	2	10·9	10	10·0
11½	12·3	15	11·6	4	10·7	12	9·6

21. *May* 17, 1786.

Barometer 30·00; Thermometer 60.

3 14	9·3	4 12	8·3	5 10	7·5	6 8	6·7
4 0	9·1	14	8·2	12	7·4	12	6·6
2	9·0	5 0	8·1	14	7·2	7 0	6·5
4	8·9	2	7·9	6 0	7·1	4	6·3
6	8·7	4	7·8	2	7·0	8	6·1
8	8·6	6	7·7	4	6·9	12	6·0
10	8·5	8	7·6	6	6·8	8 0	6·0

22. *May* 18, 1786.

Barometer 30·13; Thermometer 61.

8 4	6·0	12 0	4·8	21	3·6	44	2·45
8	5·9	13	4·6	22	3·5	48	2·35
12	5·8	14	4·5	24	3·3	52	2·25
9 0	5·7	15	4·3	26	3·2	56	2·15
8	5·5	16	4·1	28	3·1	60	2·05
10 0	5·4	17	4·0	30	3·0	64	2·0
8	5·2	18	3·9	32	2·9	68	1·9
11 0	5·1	19	3·8	36	2·7		
8	5·0	20	3·7	40	2·55		

23. *May* 22, 1786.

Barometer ; Thermometer .

72	1·9	80	1·8	100	1·65	124	1·45
74	1·85	84	1·75	108	1·6	132	1·4
76	1·85	92	1·7	116	1·5	140	1·4

After the last number, the thread broke with the 140 oz in winding up, which interrupted the experiments.

24. *May* 23, 1786.

Barometer 30·20; Thermometer 59.

Having completed the experiments with both sides of the largest hemisphere, began now the similar ones with the lead only of the same weight, the results of which are below.

Weight		Time	Weight		Time	Weight		Time	Weight		Time
oz	dr	sec.	oz	dr	sec.	oz	dr	sec.	oz	dr	sec.
1	0	19	1	6	7½	1	8	6½	1	10	5½
1	2	11	1	7	7	1	9	6	1	12	5
1	4	9									

25. *May* 24, 1786.

Barometer 30·20; Thermometer 61.

1	14	5·0	4	0	2·9	8	8	1·9	22	0	1·1
2	0	4·7	4	8	2·75	9		1·8	24		1·1
2	4	4·4	5	0	2·6	10		1·75	26		1·1
2	8	4·0	5	8	2·45	11		1·65	28		1·0
2	12	3·7	6	0	2·3	12		1·55	30		1·0
3	0	3·5	6	8	2·2	14		1·45	32		1·0
3	4	3·3	7	0	2·1	16		1·35	34		0 9
3	8	3·1	7	8	2·0	18		1·25			
3	12	3·0	8	0	1·95	20		1·2			

26. Having now obtained a long series of actuating weights and velocities, with both sides of the hemisphere foremost, and with the lead alone; a set of the corresponding ones are extracted from the whole, and arranged together, in the first of the following tables; where the first column contains the several rates of velocity, from 3 to 20 feet in a second, differing always by 1; the next 3 columns containing the correspondent weights acting on the axis, of 1·044 radius, for both sides of the body and the lead. But the differences of these not being perfectly regular, though very nearly so; correcting therefore those differences here and there, by some very small fractions, and by them the few

irregular weights, the more accurate series of weights are then arranged in the next 3 columns, which are to be considered as the true numbers. Then, subtracting the actuating weights with the lead, from each of the corresponding weights with both sides of the hemisphere, the differences are set in the next two columns, which contain the true weights acting on the axis. And finally, dividing each weight for the flat side, by the corresponding weight for the round side, the quotients are placed in the last column of the table, showing the ratio between the resistances on the two sides.

Velocity.	Irregular weights.			Regular weights.			Diffs. or true wts.		Ratios
	flat	round	lead	flat	round	lead	flat	round	
feet	oz	oz	oz	oz	oz	oz	oz	oz	
3	3·8	2·2	1·2	3·8	2·3	1·2	2·6	1·1	2·37
4	6·1	3·4	1·4	6·2	3·4	1·4	4·8	2 0	2·40
5	9·3	4·9	1·7	9·2	4·9	1·7	7·5	3·2	2·35
6	12·7	6·6	2·0	12·8	6·7	2·0	10·8	4·7	2·30
7	17	8·7	2·5	17·0	8·7	2·4	14·6	6·3	2·31
8	22	11·0	3·0	21·9	11·0	2·9	19·0	8·1	2·35
9	28	13·3	3·5	27·6	13·5	3·5	24·1	10·0	2·41
10	34	15·8	4·2	34·0	16·2	4·2	29·8	12·0	2·48
11	40	19	5·2	41·0	19·2	5·0	36 0	14·2	2·53
12	48	23	6·0	48·7	22·6	5·9	42·8	16·7	2·56
13	56	27	6·8	57·1	26·4	6·8	50·3	19·6	2·57
14	64	31	7·8	66·2	30·6	7·8	58·4	22·8	2·56
15	73	35	8·6	76·0	35·1	8·9	67·1	26·2	2·56
16	84	40	10	86·6	40·0	10·0	76·6	30·0	2·55
17	101	45	11	98·2	45·3	11·2	87·0	34·1	2·55
18	112	50	12	111	51·0	12·4	98·6	38·6	2·55
19	121	57	14	125	57·2	13·7	111·3	43·5	2·56
20	140	64	15	140	64·0	15·0	125·0	49 0	2·55

But note, that all these weights must be divided by 51·1, to reduce them to the centre of the ball.

27. By the last column it appears, that the resistance to the flat side, is to the resistance on the round side, on an average, as 2·48 to 1, or nearly as $2\frac{4}{9}$ to 1; instead of 2 to 1, as expected by the theory.—From the two columns for

both the flat and round sides, it appears, that their resistances increase in a ratio a little higher than that of the square of the velocity: but how much higher, will be determined hereafter, as well as several other inferences.

28. *June* 26, 1786.

Barom. 29·96; Thermom. ; very warm.

Having previously got prepared, for the remaining experiments, a flat disc, or very short cylinder, also a complete globe, and a cone, all of pasteboard, and of the same diameter of the preceding largest hemisphere; I proceeded to try, whether a different figure hindmost, as well as foremost, would make any difference, and what, in the results: thus, the hemisphere, the cylinder, and the cone, each with the circle foremost, have three different figures hindmost; and vice versa: also the sphere and hemisphere, with the round side foremost, have different figures hindmost; all of which circumstances will make a great variety of cases. Had the apparatus all well adjusted; the instrument nicely levelled and fixed; and had the upper pulley, over which the cord of the weight passed, fixed so high, as to have 30 or 40 turns completed, before the actuating weight could reach the floor; which would give the medium more exactly of a great number of rounds. The following numbers are the mediums, each of 30 rounds, found by subtracting the time of the 3rd round from that of the 33rd, and dividing the remainder by 30.—The cylinder was used this day, and consequently a flat side both before and behind. Its weight was 5 oz 1 dr.

Weight		Time	Weight		Time	Weight		Time	Weight		Time
oz	dr	sec	oz	dr	sec.	oz	dr	sec.	oz	dr	sec.
2	0	15·10	3	4	10·57	3	12	9·50	4	12	8·23
2	2	14·40	3	6	10·32	4	0	9·07	5	0	7·93
3	0	11·13	3	8	9·90	4	4	8·83	5	4	7·72
3	2	10·85	3	10	9·73	4	8	8·50	5	8	7·47

29. *June* 27, 1786.

Barom. 29·94; Therm. 65, at 9 A. M.

Weight		Time	Wt	Time	Wt	Time	Wt	Time
oz	dr	sec.	oz	sec.	oz	sec.	oz	sec.
5	12	7·23	9	5·60	20	3·67	64	1·97
6	0	7·05	10	5·32	22	3·48	72	1·88
6	4	6·87	11	5·02	26	3·20	80	1·80
6	8	6·67	12	4·78	30	2·97	90	1·67
7	0	6·47	13	4·58	34	2·77	100	1·60
7	8	6·23	14	4·38	40	2·53	110	1·53
8	0	6·00	16	4·10	48	2·30	120	1·50
8	8	5·90	18	3·87	56	2·10	Thread	broke.

30. *June* 28, 1786.

Barometer 30·06; Thermometer 65.

Experiments with the whole sphere; wt. 8 oz 14 dr.

2	14·5	12	3·35	30	2·07	80	1·22
3	8·55	14	3·1	34	1·95	90	1·15
4	6·75	16	2·85	40	1·8	100	1·07
5	5·75	18	2·7	48	1·65	110	1·02
6	5·1	20	2·6	56	1·5	120	0·97
8	4·3	22	2·45	64	1·4		
10	3·8	26	2·22	72	1·3		

31. The globe required the first weight, or 2 oz, just to put it in motion; and a less weight than 2 oz would not stir it at all. This was owing probably in part to the great weight of the globe, which was made of very thick pasteboard, and partly to the great length of the silk cord, between the barrel and the weight, at first to be moved, which was not less than 23 feet in length. These extra weights causing greater friction and resistance on the machine, it was to be expected that the times of revolution, with the same actuating weights, would be greater, and considerably so when those weights should be very small; but that the difference would gradually diminish as the weights increase; till at length they might become insensible, if the difference

of shape in the afterpart of the figure should not make some sensible difference; a circumstance still remaining to be tried. Accordingly, the experiments show this to be precisely the case, the difference in the times of revolution, between the globe and the hemisphere, being considerable with the small weights, but gradually and continually decreasing, so as nearly to vanish, in comparison with the whole times. So that it appears there is almost no difference, whether a flat side or a round one is hindmost, at least none that is very sensible in the swiftest of these velocities. Those that are set down above, are the mediums of the last 20 revolutions with each weight. Where it is to be noted, that, by mistake in measuring, the centre of the ball was placed $\frac{1}{10}$ of an inch too near the axis; so that the times should be a very little more than are here set down, viz, more in the proportion nearly of $53\frac{1}{4}$ to $53\frac{1}{3}$, which however will not sensibly alter them.

32. *June* 29, 1786.

Weather much the same as yesterday.

Used the cone this day. Its weight was 8 oz 6 dr, and its height $6\frac{1}{8}$ inches, being rather more than its diameter, which was $6\frac{3}{8}$, being the same diameter as the former figures. Tried it with the vortex foremost this day, as follows.

Wt.	Time	Wt.	Time	Wt.	Time	Wt.	Time
oz	sec.	oz	sec.	oz	sec.	oz	sec.
2	13·50	8	4·42	18	2·80	48	1·65
3	8·65	9	4·15	20	2·67	56	1·52
4	6·95	10	3·87	24	2·40	64	1·42
5	5·97	12	3·57	28	2·17	72	1·35
6	5·30	14	3·25	32	2·07	80	1·30
7	4·77	16	3·00	40	1·85	100	1·17

33. It was found that 2 oz was the least weight that would give motion to the machine.—By comparing these numbers with those of the day before, for the globe, it appears that the times to day are nearly equal to, but somewhat

greater than, those of the former day; and consequently, that the resistance on the convex surface of the cone, is nearly equal to, but rather greater than, that on the globe or hemisphere, the diameters being equal, but the height of the cone being a little more than double the height of the hemisphere.

34. The following experiments were next made with the base of the cone foremost.

Wt.	Time	Wt.	Time	Wt.	Time	Wt.	Time
oz	sec.	oz	sec.	oz	sec.	oz	sec.
12	4·95	24	3·40	48	2·37	72	1·93
16	4·25	30	3·05	56	2·20	80	1·85
20	3·77	40	2·60	64	2·02	100	1·65

These times are rather higher than those of the cylinder. But the cone was rather the heavier, and their after parts different.

35. *June* 30, 1786.

Barom. 30·02; Therm. 64, at 9 A. M.

Put on the smaller hemisphere, the flat side to go foremost, in order to compare with the large one, to try if the resistance be as the surface, when the axis or radius is the same. The flat surface of this hemisphere is equal to $\frac{1}{3}$ of a square foot, but that of the large one $\frac{2}{9}$, their proportion being as 16 to 9, the radius of the small one's path 52·42, of the large one 53·34; the compound ratio of the surface and path or radius, being 16 × 53·34 to 52·42 × 9, viz, 38 to 21 nearly, or as 11 to 6, but not quite so near.—Note that 6 dr just gave motion to the machine.

1	21·80	7	4·67	20	2·65	56	1·57
2	10·82	8	4·35	24	2·45	64	1·45
3	7·70	10	3·90	28	2·25	72	1·40
4	6·47	12	3·50	32	2·10	80	1·32
5	5·67	14	3·25	40	1·82		
6	5·15	16	3·00	48	1·67		

The following were with the lead only, of the same weight with this small hemisphere, viz, 2 oz 5 dr.

Wt.	Time	Wt.	Time	Wt.	Time	Wt.	Time
oz	sec.	oz	sec.	oz	sec.	oz	sec.
1	8·10	2¼	3·55	4	2·40	8	1·60
1¼	6·00	2½	3·30	4½	2·25	9	1·50
1½	4·95	2¾	3·10	5	2·10	10	1·40
1¾	4·35	3	2·90	6	1·90	11	1·30
2	3·80	3½	2·60	7	1·70	12	1·25

36. *July* 5, 1786.

Barom. 30·32; Therm. 66, at 4 P. M.

Made a thin leaden weight of 8 oz 10 dr, being only 4 dr more than the cone, and 4 dr less than the whole globe, so that it nearly agrees with both. Experiments with it as below.

Wt.		Time	Wt.		Time.	Wt.		Time	Wt.		Time.
oz	dr	sec.	oz	dr	sec.	oz	dr	sec.	oz	dr	sec.
1	8	9·0	2	8	4·2	4	0	2·8	7	0	2·0
1	10	7·0	2	10	4·0	4	4	2·7	8		1·8
1	12	6·4	2	13	3·8	4	8	2·65	9		1·7
1	14	5·9	3	0	3·6	4	12	2·6	10		1·6
2	0	5·3	3	3	3·4	5	0	2·5	11		1·5
2	2	4·9	3	6	3·2	5	8	2·4	12		1·4
2	4	4·6	3	9	3·0	6	0	2·2½	13		1·35
2	6	4·4	3	12	2·9	6	8	2·1	14		1·3

37. The latter part of the preceding numbers in the times being less than those for the less weight, on May 24, I repeated the experiments with this weight, of 4 oz 3 dr, to get the times for it more exactly. The reason of this difference is, that the arm went only 15 times round in descending on that day, but here 35 times round, the times diminishing a little, all the way to the end, and the times here are the means of the last 10 rounds out of the 35, which are much more exact.

Experiments with the wt. 4 oz 3 dr.

Weight		Time	Weight		Time	Weight		Time	Weight		Time
oz	dr	sec.	oz	dr	sec.	oz	dr	sec.	oz	dr	sec.
2	0	4·6	3	8	3·0	7		1·9	12		1·3
2	4	4·2	4	0	2·7	8		1·75	13		1·25
2	8	3·85	4	8	2·5	9		1·6	14		1·2
2	12	3·5	5		2·3	10		1·5			
3	0	3·3	6		2·1	11		1·4			

38. *July* 6, 1786.

Experiments continued, with the wt. 4 oz 3 dr.

1	4	10·8	1	6	8·4	1	8	7·0	1	12	5·8
1	5	9·8	1	7	7·6	1	10	6·4	1	14	5·3

The following with the Cone, the base foremost, to complete the set.

4	10	9·6	5	8	8·2	9		5·95	100		1·6
4	12	9·3	6	0	7·7	10		5·55	110		1·55
4	14	9·1	7		6·95	12		5·05	120		1·5
5	0	8·9	8		6·3						

When 120 oz was near the bottom, the cord broke.

39. Thus the course of the experiments being now pretty fully extended to various shapes and dimensions of figures, we may next bring into one point of view the weights for all the kinds of figures, and for all the velocities, from 3 to 20 feet per second. And first the irregular numbers just as they were experienced.

Experimented Velocities and Weights.

Veloc. per sec.	Cone, 8 oz 6 dr		Whole globe 8oz 14d	Cylinder 5oz 1dr	Great Hemisphere, 4 oz 3 dr		Small Hemis 2oz 5dr	Lead I. 8 oz 10 dr	Lead II. 4 oz 3 dr	Lead III. 2 oz 5 dr
	Vertex	base			round	flat	flat			
ft.										
3	2·9	4·7	2·8	3·5	2·2	3·8	2·4	1·4	1·3	0·9
4	4·0	6·9	3·9	6·0	3·4	6·1	3·6	1·6	1·5	1·1
5	5·4	9·9	5·2	9·0	4·9	9·2	5·1	1·9	1·8	1·4
6	7·3	13·7	7·1	12·5	6·6	12·8	7·1	2·2	2·0	1·7
7	9·5	18·0	9·2	16·8	8·7	17·0	9·5	2·6	2·4	1·9
8	12·5	22·9	11·3	21·8	11·0	22	12·0	3·1	2·8	2·3
9	15·1	29·0	14·0	27·6	13·3	28	15·1	3·5	3·3	2·7
10	18·0	35·6	16·7	33·5	15·8	34	18·3	4·0	3·8	3·1
11	21·8	41·7	20·7	39·7	19·0	40	22·0	4·4	4·4	3·6
12	25·2	49·9	24·0	47·0	23	48	26·4	5·8	5·1	4·1
13	28·8	58·2	28·0	54	27	56	30·7	6·2	5·7	4·7
14	34·5	65·8	32·0	62	31	64	34·9	7·0	6·5	5·5
15	39·3	78·0	37·2	72	35	73	38·6	7·7	7·2	6·2
16	44·0	90	42·7	84	40	84	43·7	8·5	8·0	6·8
17	48·6	101	48·5	95	45	101	50·4	9·5	8·6	7·6
18	54·2	110	53·3	109	50	112	57·3	10·5	9·4	8·5
19	60·0	126	58·4	125	57	121	62·7	11·3	10·3	9·3
20	66·3		64·0		64	140	72·0	12·0	11·0	10·0

40. But as there are some small irregularities in a few of these numbers, as appears by taking their differences; by correcting these differences so as to make them regular, and then by them correcting the few irregular actuating weights, the more correct table of those numbers will be as follows.

Table of Regular Actuating Weights and Velocities.

Veloc. per sec.	Cone, 8 oz 6 dr		Whole globe 8 oz 14 dr	Cylinder, 5 oz 1 dr	Large Hemisphere, 4 oz 3 dr		Small Hemis. 2oz 5dr	Lead I. 8 oz 10 dr	Lead II. 4 oz 3 dr	Lead III. 2 oz 5 dr
	Vertex	Base			round	flat	flat			
feet										
3	2·9	4·7	2·8	3·5	2·2	3·8	2·4	1·4	1·2	1·0
4	4·1	7·2	4·0	6·0	3·4	6·3	3·6	1·6	1·4	1·2
5	5·5	10·2	5·4	9·0	4·9	9·3	5·1	1·9	1·7	1·5
6	7·2	13·7	7·0	12·5	6·7	12·8	7·0	2·2	2·0	1·8
7	9·2	17·8	9·0	16·6	8·7	16·9	9·2	2·6	2·4	2·1
8	11·6	22·5	11·3	21·3	11·0	21·6	11·7	3·0	2·8	2·5
9	14·3	27·9	14·0	26·7	13·5	27·0	14·6	3·5	3·3	2·9
10	17·3	34·0	17·0	32·8	16·2	33·1	17·8	4·0	3·8	3·4
11	20·7	41·0	20·4	39·6	19·2	40·0	21·4	4·6	4·8	3·9
12	24·4	48·6	24·1	47·2	22·6	47·6	25·4	5·2	4·9	4·4
13	28·4	57·0	28·1	55·6	26·4	56·0	29·7	5·9	5·5	5·0
14	32·8	66·2	32·4	64·8	30·6	65 2	34·4	6·6	6·2	5·6
15	37·5	76·2	37·1	74·8	35·1	75·2	39·4	7·4	6·9	6·2
16	42·6	87·2	42·1	85·8	40·0	86·2	44·8	8·2	7·6	6·9
17	48·0	99·2	47·5	97·8	45·3	98·2	50·5	9·1	8·4	7·6
18	53·8	112·3	53·3	110·9	51·0	111·3	56·6	10·0	9·2	8·4
19	60·0	126·5	59·5	125·1	57·2	125·5	63·1	11·0	10·1	9·2
20	66·6	141·8	66·0	140·4	63·8	140·8	70·1	12·0	11·0	10·0

41. Then, by subtracting the numbers in the last three columns of the leads, from those of their corresponding bodies, the remainders will show the regular set of resistances to the respective figures, as in the following table.—The column of numbers to be subtracted from that of the cylinder, which weighed 5 oz 1 dr, was deduced, by proportion, from the last column but one of the preceding table, which has the lead nearest of the same weight with it.

Table of Regular Mean Resistances to several Figures.

Veloc. per sec.	Small Hemis. flat	Cone.		Whole Globe	Cylinder	Large Hemis.		Ratio of flat to round
		Vertex	Base			convex	flat side	
3	1·4	1·5	3·3	1·4	2·4	1·1	2·6	2·36
4	2·4	2·5	5·6	2·4	4·6	2·1	4·9	2·33
5	3·6	3·6	8·3	3·5	7·3	3·2	7·6	2·38
6	5·2	5·0	11·5	4·8	10·5	4·7	10·8	2·30
7	7·1	6·6	15·2	6·4	14·2	6·3	14·5	2·30
8	9·2	8·6	19·5	8·3	18·4	8·2	18·8	2·30
9	11·7	10·8	24·4	10·5	23·3	10·2	23·7	2·32
10	14·4	13·3	30·0	13·0	28·9	12·4	29·3	2·36
11	17·5	16·1	36·4	15·8	35·2	14·9	35·7	2·40
12	21·0	19·2	43·4	18·9	42·2	17·7	42·7	2·41
13	24·7	22·5	51·1	22·2	50·0	20·9	50·5	2·42
14	28·8	26·2	59·6	25·8	58·5	24·4	59·0	2·42
15	33·2	30·1	68·8	29·7	67·8	28·2	68·3	2·42
16	37·9	34·4	79·0	33·9	78·0	32·4	78·6	2·43
17	42·9	38·9	90·1	38·4	89·2	36·9	89·8	2·43
18	48·2	43·8	102·3	43·3	101·5	41·8	102·1	2·44
19	53·9	49·0	115·5	48·5	114·8	47·1	115·4	2·45
20	60·1	54·6	129·8	54·0	129·2	52·8	129·8	2·46

Here, besides the columns of experimented resistances, in avoirdupois ounces and tenths, for each figure; another column is added at the end, showing the ratios between the correspondent numbers in the two next preceding columns, being the resistances for the round and flat sides of the large hemisphere; which ratios are found by dividing always the greater numbers by the less; showing that they proceed always increasing a very little, as the velocity is increased; the medium among all these being 2·4, instead of 2 only, as expected by the theory of resistances to such figures. But as part of this ratio may be owing to the different shape of the after or hinder sides of the figures, the more correct comparison would be between either the whole sphere and the flat side of the large hemisphere, or the cylinder and the round side of the same large hemisphere, in both of which cases the after parts would be of the same shape, and then

the medium ratio would be near 2·3 to 1.—After all, it is to be observed, that the numbers in the preceding table denote, not the true measure of the resistances, but only the weights acting at the circumference of the barrel or axis, while the real resistance of the air acts on the body at the extremity of the arm. To obtain the true resistances then, those numbers must be all divided by the ratios of the corresponding radii or distances at which the weight and the resistance act, viz, by 50·2 for the small hemisphere, and by 51·1 for all the other figures, as determined on the 2nd day of May; this being done, the true measure of the resistances will be as in the table below, for all the velocities from 3 to 20 feet per second of time.

42. *Table of the true Resistances to the Figures in Ounces.*

Veloc. per sec.	Cone.		Whole globe	Cylinder	Hemisphere.		Small Hemis. flat	Index of the power of the velocity.
	Vertex	Base			flat	round		
feet	oz	oz	oz	oz	oz	oz	oz	
3	·028	·064	·027	·050	·051	·020	·028	
4	·048	·109	·047	·090	·096	·039	·048	
5	·071	·162	·068	·143	·148	·063	·072	
6	·098	·225	·094	·205	·211	·092	·103	
7	·129	·298	·125	·278	·284	·123	·141	
8	·168	·382	·162	·360	·368	·160	·184	
9	·211	·478	·205	·456	·464	·199	·233	
10	·260	·587	·255	·565	·573	·242	·287	
11	·315	·712	·310	·688	·698	·297	·349	2·049
12	·376	·850	·370	·826	·836	·347	·418	2·042
13	·440	1·000	·435	·979	·988	·409	·492	2·036
14	·512	1·166	·505	1·145	1·154	·473	·573	2·031
15	·589	1·346	·581	1·327	1·336	·552	·661	2·031
16	·673	1·546	·663	1·526	1·538	·634	·754	2·033
17	·762	1·763	·752	1·745	1·757	·722	·853	2·038
18	·858	2·002	·848	1·986	1·998	·818	·959	2·044
19	·959	2·260	·949	2·246	2·258	·922	1·073	2·047
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196	2·051
Propor. Numb.	126	291	124	285	288	119	140	2·040

Where the numbers placed at the bottom of all the columns, denote the general mean proportions of their resistances to one another, from the velocity 10, to each of the following ones.

Note. The general mean height of the barometer was 30·1, and of the thermometer 62°; in which circumstances therefore, the ratio of the weight of water to air, was as 840 to 1, or the cubic foot of air weighed nearly $1\frac{1}{5}$ oz avoirdupois.

It may be repeated also, that the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base, or great circle, was $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly. But the common diameter of all the other figures was $6\frac{3}{8}$ inches, and therefore the area of the circular base, was just 32 square inches, or $\frac{2}{9}$ of a square foot. Also, the altitude of the cone was $6\frac{5}{8}$ inches; its side is therefore inclined to the axis in an angle of 25° 42′.

43. From the preceding table of resistances, several practical inferences may be drawn; such as the following.

(1). That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and 8th columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion; and the mean resistances are as 140 to 288, or as 5 to $10\frac{2}{7}$, so far as these experiments have been extended. Nearly the same proportions result from the comparison of the numbers in other columns with similar surfaces. Now, the ratio of the surfaces being that of 5 to 9; and the ratio of the resistances as 5 to $10\frac{2}{7}$; it appears that the latter exceeds the former in the ratio of 9 to $10\frac{2}{7}$, or of 1 to $1\frac{1}{7}$; or of 7 to 8. Hence then, to find the resistance to any surface, from hav-

ing given that of a similar smaller one, with an equal velocity; first raise the given resistance in the proportion of the less surface to the greater; then increase the result by $\frac{1}{7}$ part, or in the ratio of 7 to 8, which will give the resistance for the larger surface, nearly. And, reverse wise, to find the resistance on a smaller body, from that of a like larger one, with equal velocity.—We shall afterwards enquire whether this proportion will answer in very high velocities also.

(2). The resistance to the same surface, with different velocities, is, in these slow motions, nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This will appear, on trial, from all the columns; and the index of the power of the velocity is set down in the last column, for the resistances in the 4th column, for the whole globe, by comparing that for the velocity 10, with each of the following ones, 11, 12, 13, &c; where, after the first three or four, the numbers gradually and slowly increase, and would doubtless continue to do so to a very great extent. The medium among those here set down, is 2·040; whence it appears, that in these slow motions, the resistance to the same body, is nearly as the 2·04 power of the velocity, or indeed as the square of it nearly.

An easy method of determining those indices, is as follows: Make a small tablet, as in the margin, placing, in the first column, the velocities from 10; in the 2nd column the correspondent resistances from any column of the former table, as suppose those for the whole globe. Then, to fill up the 3rd column, put $r=255$ the first resistance, r' = any other, as the 2nd 310; and x = the required index of the power of the velocities, which shall be proportional to the resistances; that is, so as

Veloc	Resist.	Index
10	255	
11	310	2·049
12	370	2·042
13	435	2·036
14	505	2·031
15	581	2·031
16	663	2·033
17	752	2·038
18	848	2·044
19	949	2·047
20	1057	2·051

that $10^x : 11^x :: r : r'$. Dividing the first two terms by 10, and the latter two by r, the proportion becomes $1 : 1{\cdot}1^x :: 1 : \frac{r'}{r}$; consequently $1{\cdot}1^x = \frac{r'}{r}$; the logarithm of this is $x \times \log. 1{\cdot}1 = \log. r' - \log. r$; and hence $x = \frac{\log. r' - \log. r}{\log. 1{\cdot}1} = 2{\cdot}049$, by using the aforesaid numbers for r and r'. In like manner, taking, in the denominator of the general expression for x, the successive numbers 1·2, 1·3, 1·4, &c, to 2·0; and in the numerator the successive values of r', viz, 370, 435, &c, that of r being constantly $= 255$; all the numbers in the 8rd column will be easily found.

(3.) The round ends, and sharp ends, of solids, suffer less resistance than the flat or plane ends, of the same diameter: thus, the cylinder, and the flat ends of the hemisphere and cone, have more resistance than the round or sharp ends of the same. But the sharper end has not always the smaller resistance: thus, the round side of the hemisphere has less resistance than the pointed end of the cone.

(4.) When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts should be exactly alike and equal; owing probably to the different pressures of the air, on the after parts. Thus, the resistance to the fore part of the cylinder, is less than that on the equal flat surface of the cone, or of the hemisphere; because the hind part of the cylinder is more pressed or pushed forward, by the following air, than those of the other two figures: also, for the same reason, the base of the hemisphere shows a less resistance than that of the cone, and the round side of the hemisphere less than the whole sphere.

(5.) The resistance on the base of the hemisphere, is to that on the convex side, or on the whole sphere, nearly as $2\frac{2}{5}$ to 1, instead of 2 to 1, as assigned by theory. And the experimented resistance, in each of these cases, is nearly $\frac{1}{4}$ part more than that which is determined by the theory.

(6.) The resistance on the base of the cone, is to that on the vertex, nearly as 2·3 to 1. And almost in the same ratio is radius to the sine of the angle of inclination of the side of

the cone to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same.

(7). Hence may be determined what will be the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let a = the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;

r = the resistance to the velocity in the table; and

x = the altitude sought, of a column of air, whose base is a, and its pressure r.

Then ax = the content of the column in feet, and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet, namely, $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is nearly as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then, $x = \frac{5}{6} \times \frac{r}{a}$, becomes $x = \frac{15}{4}r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634 oz to a velocity of 16 feet per second, then r = ·634, and $x = \frac{15}{4}r$ = 2·3775 feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance $r = 1{\cdot}526$: then $x = \frac{15}{4} r = 5{\cdot}72$; which exceeds the height, 4, due to the velocity, in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger, and also if the velocity were more.

(8). Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area being $\frac{2}{9}$ of a foot, is ·051 oz, with the velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056 oz only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz, is the whole pressure of the atmosphere. Therefore, as $\sqrt{{\cdot}0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

(9). Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb to such a velocity!

(10). Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen, when the ball moves faster than the particles of air can follow, by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent

air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 13{\cdot}6 \times 833\frac{1}{3}$ or 28333 feet; therefore $\sqrt{16} : \sqrt{28333} :: 32 : 8\sqrt{28333} = 1346$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1346 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its [illegible] part, as well as the resistance arising from the *vis* [illegible] of the particles of air struck by the ball.

(11). On a review of the whole of the premises, we find that the resistance of the air, as determined by the foregoing experiments, differs very widely, both in respect to its quantity on all figures, and in regard to the proportions of its action on oblique surfaces, from the same actions and resistances as assigned by the most plausible and imposing theories, which have heretofore been delivered, and confided in by philosophers. Hence it may be concluded, that all the speculative theories on the resistance of the air, hitherto laid down, are very erroneous; and that it is from experiments, carefully and skilfully executed, that a rational hope can be grounded, of deducing and establishing a true and useful theory of the actions of forces, so intimately connected with the numerous and important concerns in human life.

Proceed we now to relate the further progress of our experiments with the whirling-machine, on other bodies, and in other positions.

EXPERIMENTS OF THE YEARS 1787 and 1788.

July 31, 1787.

44. By comparing the experimented resistance on the vertex of the cone, with the theory, it appears that it is nearly equal to the resistance of the sphere; whereas, by the received theory, it ought to be only about $\frac{2}{3}$ of that of the sphere. I therefore this day tried these two bodies again, and found the results nearly the same as before. With the 32 oz weight, the mediums among the times of a revolution, were thus: viz,

This year they were	[illegible]t year they were
for the cone 2·10″	[illegible]
for the globe 2·02	2·01, nearly the same.

So that it seems difficult to account for so great a difference between theory and experiment, in the case of the cone's vertex.

Aug. 5, 1787.

Fine, clear, warm Weather.

45. Performed experiments with the following bodies, and found the medium of the times of revolution as follows.

Large Globe	The Cone.		Samll Hemis.		Lead of 2 oz 5 dr
	vertex	base	base	conv.	
2″	2·05″	2·9″	2·3	1·6	1·6

The actuating weight was 32 oz for each figure, and 8 oz for the thin lead weight, the same as June 30, 1786.

EXPERIMENTS IN 1788 WITH THE WHIRLING-MACHINE.

July 23, 1788.

46. Prepared the machine to make experiments with figures of shapes different from the foregoing ones. Providing a thin rectangular plate of brass, to fix on the arm of the machine; its weight 11 oz $0\frac{3}{4}$ dr, or $11\frac{1}{27}$ oz, and its dimensions 8 inches by 4, consequently its area was 32 square inches, the same as the plane surface of the cone and large hemisphere, before employed. It was adapted for fitting on the end of the arm in both directions, viz, in the direction both of the length and of the breadth, to try the effect of the same surface in both positions. It was also contrived to incline the surface in any degree to the direction of motion, to try the resistance at all angles of inclination. When fitted on with its length in the direction of the arm, the distance of its centre, from the axis of motion, was $53\frac{3}{4}$ inches; and the same distance also when fitted on the other way.

July 24, 1788.

Barometer 30·0; Thermometer $70\frac{1}{2}$.

47. Tried the experiments this day with the plane fitted on with its length in the direction of the arm, its centre being distant $53\frac{3}{4}$ inches from the axis of motion. The plane was inclined in various angles, as expressed below; and the actuating weight was 32 oz or 2lb each time.

Table of the Times of Revolution, the Plane lengthwise.

0° or horizontal	5°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90° or vertical
1·00″	1·17″	1·34″	1·67″	1·98″	2·27″	2·40″	2·54″	2·78″	2·94″	3·00″	3·02″

In the first line of this table, are the angles in which the plane was inclined to the horizon, or to its path, sometimes at 5 degrees, and sometimes at 10°, difference from each

other; the first position being 0°, or horizontal, with its edge to the air; and the last 90°, or the plane vertical, with its face directly opposed to the path or resistance of the air. In the second line are the intervals of time in performing each revolution, expressed in seconds and hundredth parts; each number, as usual, being a medium of several repetitions.

Note, The area of the plane is 32 square inches, or $\frac{2}{9}$ sq. foot, and = the circle of the large hemisphere.

The radius of revolution to centre of the plane $53\frac{3}{4}$ inches.

Circumference described by its centre in 1 revolution 337·7 inc. or 28·14 feet.

Ratio of radius of axis to radius of revolution, as 1 to 51·51, accounting the radius of axis to the middle of the thread, 1·044 inches.

July 25, 1788.

Barometer 60·10; Thermometer 66.

48. Used the same brass plane again, set at various angles as before, but fixed on the contrary way, viz, the breadth or shorter dimension in the direction of the arm; but still having its centre at the same distance of $53\frac{3}{4}$ inches from the centre of motion; to try if there be any difference by such change of position; the actuating weight being still the same 32 ounces.

Times of Revolution, with the Plane crosswise.

0° or horiz.	5°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90° or vert.
1·00″	1·17″	1·35″	1·69″	2·00″	2·30″		2·57″	2·80″	2·93″	3·00″	3·03″

Here, as before, the first line contains the inclination of the plane to the direction of motion, and the second the medium times of revolution. The inclination of 45° was not used this day; but it was tried in two different ways the day before, viz, inclined 45° backwards, thus ⟍, and then inclined 45° forward, thus ⟋, when the result was the same in both cases: whence it may be concluded, that the resistance is the same, whether the plane is inclined forward or backward, at each angle of inclination, as indeed it might be expected.

By comparing the experiments of this day, with those of the day before, it appears that the differences are very small indeed; but those of this day are rather longer intervals, if any thing; but the difference is so small, and doubtful, that the times may be accounted equal.

49. After the above experiments, the following course was then instituted with a thin bit of lead, on the end of the arm, equal to the weight of the brass plane, and which, its resistance being very small, will give nearly the weights to be deducted from all the above, on account of the friction of the machine and the resistance on the arm, viz, with the

actuating wts. in oz.	8	10	12	16	20	24	28	32
time of each revolut.	3·33″	2·50	2·00	1·54	1·35	1·20	1·09	1·00

And from these, by interpolation are deduced the following.

Incl.	0	5°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Wts.	oz. 32	24·8	19·5	15·5	12·7	10·9	9·9	9·3	8·9	8·7	8·6
Times	1·00′	1·17″	1·35″	1·68″	1·99″	2·28″	2·56″	2·79′	2·94″	3·00″	3·02″

50. And hence the following general table.

Inclin. of the plane.	Time of one revolution.	Actuating wt. with plane	Actuating wt. without plane	Diff. or resistance.	Ditto reduced to arm.	Velocity per second.
		oz	oz	oz	oz	feet
0°	1·00″	32	32·0	0·0	·000	28·1
5	1·17	32	24·8	7·2	·140	24·1
10	1·35	32	19·5	12·5	·243	20·8
20	1·68	32	15·5	16·5	·320	16·8
30	1·99	32	12·7	19·3	·371	14·1
40	2·28	32	10 9	21·1	·410	12 4
45	2·42	32	10·4	21·6	·420	11·6
50	2·56	32	9·9	22·1	·429	11·0
60	2·79	32	9·3	22·7	·441	10·1
70	2·94	32	8·9	23·1	·448	9·6
80	3·00	32	8·7	23·3	·452	9·4
90	3·02	32	8·6	23·4	·454	9·3
1	2	3	4	5	6	7

Here, the numbers in the 6th column, in thousandth parts of an ounce, are equal to the resistance of the plane, placed at the angle on the same line in the first column, when making one revolution, or describing the space $28\frac{1}{7}$ feet, in the time or number of seconds in the 2nd column, or when moving at the rate denoted by the numbers in the 7th column.

From what has been done, in the last two days, it appears, that there is no sensible difference between the two different ways of putting on the plane, viz, the length or breadth in direction of the length of the arm; provided its centre be but at the same distance from the axis, in both cases.

But now to obtain the law of resistance, for the several angles of inclination, with one and the same velocity, it was proper to institute a fresh set of experiments, as they here follow.

July 31, 1788.

Barometer 29·91; Thermometer 72.

51. The object of this day's experiments was, to find the resistance on the same plane, at every angle of elevation, when the velocity, as for example, is 20 feet per second. I therefore began as follows, putting on different weights, to bring the velocity to about 20 feet in a second.

Angle with horiz.	Actuating wts.	Time of one revolution.	Veloc. per second.
	oz	sec.	feet
0°	19	1·50	18·76
·	20	1·43	19·68
5	22	1·43	19·68
·	23	1·33	21·16
10	24	1·43	19·68
·	25	1·40	20·10
20	26	1·76	16·00
·	27	1·73	16·27
·	28	1·67	16·85
·	29	1·65	17·06
·	30	1·58	17·81
·	32	1·56	18·04
·	34	1·52	18·51
·	36	1·48	19·02
·	38	1·44	19·54
·	40	1·36	20·69
30	44	1·60	17·59
·	48	1·54	18·27
·	52	1·43	19·68
·	56	1·40	20·10
40	64	thread	broke

The thread having broken with the last weight, of 64 oz, the experiments could not be further prosecuted for the proposed velocity, 20 feet. I was therefore obliged to be content with the case of a smaller velocity, as that of 12 feet for example, which would not require so large an actuating weight; the results of which are as follow.

Angle with horiz.	Actuating wts.	Time of 1 revol.	Veloc. per sec.	Diffs. of wts.	Resist. to 12 ft. veloc.	Ditto reduced.	Ratio to the last.	Sines of the angles
	oz	sec	feet	oz	oz	oz		
0°	10	2·50	11·25	0	0	0	0	0
5	$10\frac{1}{4}$	2·67	10·54	$0\frac{1}{4}$	0·81	·016	19	87
·	$10\frac{3}{4}$	2·43	11·58	$0\frac{3}{4}$				
10	12	2·50	11·25	2	2·28	·044	52	173
20	15	2·73	10·31	5	6·84	·133	158	342
·	16	2·50	11·25	6				
30	24	2·37	11·87	14				
·	26	2·26	12·45	16	14·31	·278	331	500
·	28	2·19	12·85	18				
40	32	2·40	11·72	22				
·	34	2·30	12·23	24	23·09	·448	533	643
·	36	2·26	12·45	26				
50	39	2·40	11·72	29				
·	40	2·39	11·77	30	31·34	·609	724	766
·	41	2·38	11·83	31				
60	43	2·50	11·25	33	37·60	·730	868	866
70	44	2·58	10·91	34	41·49	·805	957	940
·	45	2·55	11·04	35				
80	46	2·60	10·88	36	43·06	·836	994	985
·	47	2·52	11·14	37				
90	48	2·50	11·25	38	43·35	·841	1000	1000
1	2	3	4	5	6	7	8	9

52. Having, by the mediums of several trials, found the times corresponding to each revolution, in the 3rd column of this table, corresponding to the weights in the 2nd column, and angles of inclination in the first column; the circumference of revolution, viz, $28\frac{1}{7}$ feet, being divided by these times, gives the velocity per second, as placed in the 4th column.

Then the first weight, of 10 oz, when the very thin edge of the plane only was opposed to the air, being deducted from every weight in the 2nd column, leaves the weights in the 5th column, which are to be considered as the actuating weights at the barrel of the axis, very nearly.

In the 6th column are the several resistances to a velocity of 12 feet per second, as deduced from those in the 4th column, to which number these are nearly equal, by stating the resistances to be proportional to the 2·04 power of the velocity, agreeably to the rate we have before determined for slow motions, that are so nearly equal to each other.

In the 7th column, the resistances of the 6th column are reduced, in the proportion of the distance of the plane to the distance of the actuating weight; and these are the true measures of the resistances, in thousandth parts of an ounce, to the velocity 12 feet in a second of time, for each position of the plane.

In the 8th column are set the ratios of all these resistances, to the last or greatest of them; which are obtained by dividing each of them by that greatest; being done in order to compare them with the sines of the same angles of inclination, as placed in the 9th or last column Whence it appears, that the resistance is in no wise proportioned to those sines: for, from the direct vertical position, or 90 degrees, the sine of the angle of inclination decreases faster than the resistance, till at the angle of 60° inclination, where they are again equal. After which, the resistance decreases faster than the sines; but no where however so fast as the squares of the sines, and much less of their cubes. So that the resistance, it appears, is in practice not as any single power of the sines, neither 1st, nor 2nd, nor 3rd power of them.

53. But, if we would search out some powers or function of the sines, that should be proportional to the resistances, it is manifest that the exponent must be some variable quantity, which at 60° must be $= 1$, or nearly so; below 60°, it must be greater than 1, but less than 2, and above 60° it must be between 1 and 0. Now the cosines of the angles increase from 90° down to 0, in such sort, that their doubles have the above requisite property, viz, that at 60°, $2c$ is $=1$; above 60°, $2c$ is between 1 and 0; and below 60°, $2c$ is between 1 and 2; where c denotes the cosine of the angle of inclination. Let us compute then the proportions of s^{2c} for

the several angles of inclination, where s denotes the sine, in order to compare with the resistances, and we shall find the values of s^{2c}, for the several angles, to be thus:

Angles	Values of s^{2c}	Exper. Resist.	Diffs.
5°	8	19	11
10	32	52	20
20	133	158	25
30	302	331	29
40	508	533	25
50	710	724	14
60	866	868	2
70	958	957	-1
80	995	994	-1
90	1000	1000	0

Here it appears, that the values of s^{2c} agree with the ratios of the resistances, from 90° to 60° very nearly; but that from 60° to 30°, the differences increase, and from 30° to 0, they decrease again, the greatest difference being at 30°, where it is $\frac{1}{11}$ of the whole.

54. Instead of the index $2c$, in the function then, it is likely that we shall succeed better if we assume the indefinite one nc, by adapting its result to the resistance for the experimented angle of 30°, that is, to ascertain what n must be, so as that the function s^{nc} may agree with the resistance at 30°, viz, by making $s^{nc} = 331$. Now this equation in logarithms is $nc \times \log. s = \log. 331$, and hence $n = \frac{\log. 331}{c \times \log. s}$ $= 1{\cdot}842$ nearly. Then taking always $s^{1{\cdot}842c}$, there results the following numbers:

Angles.	Values of $s^{1{\cdot}842c}$	Exper. resist.	Diffs.
5°	11	18	7
10	42	52	10
20	156	158	2
30	331	331	0
40	536	533	-3
50	730	724	-6
60	876	868	-8
70	962	957	-5
80	995	994	-1
90	1000	1000	0

Here the differences being all very small, it may be concluded that the function $s^{1{\cdot}842c}$ is an expression for the resistance, sufficiently near the truth, for all the angles of inclination. And this being multiplied by ·841, to bring it to ounces, will give $\cdot 841 s^{1{\cdot}842c}$ for the resistance, in ounces, for any angle, whose sine is s, and cosine c, for the velocity of 12 feet in a second, on a plane surface of 32 square inches, or $\frac{2}{9}$ of a square foot.

Aug. 11, 1788.

Barometer 30·08; Thermometer 60.

55. The occasion of this day's experiments was as follows: On comparing the last experiments, made with the parallelogram inclined in various angles, with those formerly made with the cone, the vertex foremost, the surface of which I thought might be considered as expanded or unrolled in a triangle of an equal area, and inclined in the same angle. But the resistances not turning out to be in the proportion of the two surfaces, I imagined that the difference might result from the diversity in the shape of the two figures, or surfaces compared, the one being a plane rectangle, and the other a conical surface, or as it were a triangle rolled up.

56. I therefore procured an equilateral triangular brass plane, nearly of the same area and weight as the parallelogram last used, viz, the triangle being $31\frac{2}{7}$ square inches, or nearly $\frac{2}{9}$ of a square foot, and performed the following experiments with it, inclining the plane to the direction of motion in all angles, differing by 5° at each time, and at each inclination turning the vertex of the triangle both forewards and backwards, so that it might go both foremost and hindmost; but, as there was no difference in the resistance between the two positions, the numbers are set down only for one of them, the same numbers serving for both. The middle line of the plane was placed at $53\frac{3}{4}$ inches from the centre of motion; and therefore the velocity of that line, is to the velocity of the actuating weight, as 51·51 to 1. The weights, for each inclination, were gradually changed, so as to bring the plane to make 2 uniform revolutions in every 5 seconds of time, in which case the velocity of the plane was $11\frac{1}{4}$ feet per second, or rather 11·26 feet.

57. Having therefore placed the several angles in the first column of the following table, and the actuating weights in the 2nd column, which gave 2 revolutions in 5 seconds, in the 3rd column are set the differences between the first weight, when horizontal, and each of the other following

weights, which remainders denote the resistances without the friction and resistance on the arm, for a velocity of 11·26 feet. Then these are raised in the ratio of $11{\cdot}26^{2{\cdot}04}$ to $12^{2{\cdot}04}$, or 7 to 8 nearly, for the resistances to 12 feet velocity, which are placed in the 4th column. These last numbers are again increased in the ratio of the surface $31\frac{2}{7}$ of this triangle, to 32 the surface of the former plane, that is, in the ratio of 43 to 44, or 1 to $1\frac{1}{43}$, which increased numbers are set in the next or 5th column. Then—

Angles with direc.	Actuating wts.	Diffs. of ditto	Ditto for 12 feet velocity	Ditto for 32 inches area	Resist. reduced	Ratios to the last	Sines of the angles
	oz	oz	oz	oz	oz		
0°	10	0·0	0·0	0·0	·000	·000	·000
5	10·7	0·7	0·8	0·8	·015	·018	·087
10	11·8	1·8	2·0	2·0	·039	·046	·174
15	13·4	3·4	3·9	4·0	·077	·091	·259
20	15·8	5·8	6·6	6·8	·132	·156	·342
25	19·0	9·0	10·3	10·5	·204	·241	·423
30	23·0	13·0	14·8	15·1	·294	·347	·500
35	27·3	17·3	19·7	20·1	·390	·461	·574
40	31·6	21·6	24·7	25·2	·490	·579	·643
45	35·4	25·4	29·0	29·7	·577	·682	·707
50	38·6	28·6	32·7	33·5	·650	·768	·766
55	41·1	31·1	35·5	36·3	·705	·833	·819
60	43·0	33·0	37·7	38·6	·750	·886	·866
65	44·5	34·5	39·4	40·3	·783	·925	·906
70	45·7	35·7	40·8	41·8	·809	·956	·940
75	46·4	36·4	41·6	42·6	·827	·977	·966
80	46·9	36·9	42·1	43·1	·837	·989	·985
85	47·2	37·2	42·5	43·5	·845	·998	·996
90	47·3	37·3	42·6	43·6	·846	1·000	1·000
1	2	3	4	5	6	7	8

these are divided by 51·51, to reduce the weights from the barrel to the extremity of the arm, and the quotients are the true measures of the resistances, which are placed in the next column 6.—Lastly, each of these being divided by the greatest, 846, give their proportions in numbers easily to be compared with the sines of the angles: and these proportionals are placed in the 7th column; having the correspond-

ent sines of the angles of inclination set in the 8th or last column.

On a slight comparison it soon appears, that the numbers for the triangular shape, are nearly the same as those for the parallelogram, of equal area; evincing that the difference in the shape of the plane makes no material difference in the resistance, the area being equal.

58. By interpolation from the foregoing numbers, the following table is deduced, for every single degree of inclination, accompanied with the correspondent sines of the same angles.

Angle	Ratios	Sines	Angl.	Ratios	Sines	Angl.	Ratios	Sines
1°	3	17	31°	369	515	61°	895	875
2	6	35	32	392	530	62	903	883
3	10	52	33	415	545	63	911	891
4	14	70	34	438	559	64	918	899
5	18	87	35	461	574	65	925	906
6	23	105	36	484	588	66	932	914
7	28	122	37	507	602	67	938	921
8	34	139	38	531	616	68	944	927
9	40	156	39	555	629	69	950	934
10	46	174	40	579	643	70	956	940
11	53	191	41	602	656	71	961	946
12	61	208	42	623	669	72	966	951
13	70	225	43	643	682	73	970	956
14	80	242	44	663	695	74	974	961
15	91	259	45	682	707	75	977	966
16	102	276	46	700	719	76	980	970
17	114	292	47	718	731	77	983	974
18	127	309	48	735	743	78	985	978
19	141	326	49	752	755	79	987	982
20	156	343	50	768	766	80	989	985
21	171	358	51	783	777	81	991	988
22	187	375	52	797	788	82	993	990
23	204	391	53	810	799	83	995	993
24	222	407	54	822	809	84	997	995
25	241	423	55	833	819	85	998	996
26	261	438	56	845	829	86	999	998
27	282	454	57	856	839	87	999	999
28	303	469	58	866	848	88	999	999
29	325	481	59	876	857	89	1000	1000
30	347	500	60	886	866	90	1000	1000

Where the numbers in the 2nd column, when multiplied by 84, and divided by 1000, will give the resistance in decimals of an ounce, to a plane of 32 square inches area, moved with the velocity of 12 feet in a second of time, meeting the air in the several angles shown in the first column of the table: and the general formula for every case of that plane, as before, is $84s^{nc}$, where s denotes the sine, and c the cosine of the angle of inclination, also $n = 1{\cdot}842$, or $1{\cdot}84$ only. For other surfaces and velocities, the formula will vary nearly as the surfaces, or but little higher, about $\frac{1}{7}$, and nearly as the $2{\cdot}04$ power of the velocity. Or, for any other greater plane a, and velocity v, the formula will be nearly $\cdot 03av^{2\cdot 04}s^{1\cdot 84c}$ feet.—For water, or any other fluid, different from air, this theorem will be varied in proportion to the density of the fluid.

59. Though the figure of the plane makes no sensible difference in the resistance; yet the action on a plane will not apply to such curve surfaces as those of a cone or a sphere, as clearly appears by the table of experiments in page 189; where the convexity of the hemisphere, on a surface equal to double its base, has less than half the resistance; and the cone, with a surface of nearly 74 square inches, at an angle of 25°42′, with the path, suffers a resistance far less in proportion to the plane at an equal angle; for, the ratio of the surfaces is that of 74 to 32, much more than double, while that of the resistances, 376 to 255, is less than 3 to 2.

TRACT XXXVII.

THEORY AND PRACTICE OF GUNNERY, AS DEPENDENT ON THE RESISTANCE OF THE AIR.

SECT. 1. In this tract it is proposed to deduce the general conclusions, from the long series of experiments made in Tracts 34 and 36, and to make applications of them to various practical uses, particularly to the theory and practice of gunnery, as dependent on, or modified by the resistance of the air, now that its effects have, for the first time, been in part satisfactorily ascertained, by very numerous and accurate experiments.

2. In some parts of my Course of Mathematics, certain notices are to be found concerning the subject, both as connected with the air's resistance, and as independent of it. Thus, in the 2nd volume of that course, in props. 19, 20, 21, 22, is delivered all that relates to the parabolic theory of projectiles, that is, the principles and conclusions which would obtain, and regulate such projects, if they were not impeded and disturbed in their motions, by the air in which they move. For, from the enormous resistance of that medium it happens, that many military projectiles, especially the smaller balls discharged with high velocities, do not range so far as a 20th part of what they would naturally do in empty space. That theory then can only be useful in some few cases of slow motions, not exceeding the velocities of 200, or 300, or 400 feet in a second of time, when the path of the projectile differs but little from the curve of a parabola. Also, at pa. 160, &c, are given several other practical rules and calculations, depending partly on the foregoing parabolic theory, and partly on the results of certain experiments performed with cannon balls.

Again, in prop. 58, pa. 219, are delivered the theory and calculations of a beautiful military experiment, invented by Mr. Robins, for determining the true degree of velocity with which balls are discharged from guns, with any charges of powder. The idea of this experiment, is simply, that the ball is discharged into a very large but moveable block of wood, whose small velocity, in consequence of that blow, can be easily observed and accurately measured. Then, from this small velocity, thus obtained, the large one of the ball is immediately derived by this simple proportion, viz, as the weight of the ball, is to the sum of the weights of the ball and the block, so is the observed velocity of the last, to a 4th proportional, which is the velocity of the ball sought.—It is evident that this simple mode of experiment will be the source of numerous useful principles, as results derived from the experiments thus made, with all lengths and sizes of guns, with all kinds and sizes of balls and other shot, and with all the various sorts and quantities of gunpowder; in short, the experiment will supply answers to all enquiries in projectiles, excepting the extent of their ranges; for it will even determine the resistance of the air, by causing the ball to strike the block of wood at different distances from the gun, thus showing the velocity lost by passing through those different spaces of air; all which circumstances are fully shown in the foregoing 34th Tract.

3. Lastly, in prob. 17 on Forces, near the end of that volume, some results of the same kind of experiment are successfully applied, to determine the curious circumstances of the first force or elasticity of the air resulting from the fired gunpowder, and the velocity with which it expands itself. These are circumstances which have never before been determined with any precision. Mr. Robins, and other authors, it may be said, have only guessed at, rather than determined them. That ingenious philosopher, by a simple experiment, truly showed that, by the firing of a parcel of gunpowder, a quantity of elastic air was disengaged, which, when confined in the space only occupied by the powder before it was fired,

was found to be near 250 times stronger than the weight or elasticity of the common atmospheric air. He then heated the same parcel of air to the degree of red hot iron, and found it in that temperature to be about 4 times as strong as before; whence he inferred, that the first strength of the inflamed fluid, must be nearly 1000 times the pressure of the atmosphere. But this was merely guessing at the degree of heat in the inflamed fluid, and consequently of its first strength, both which in fact are found to be much greater. It is true that this assumed degree of strength accorded pretty well with that author's experiments; but this seeming agreement, it might easily be shown, could only be owing to the inaccuracy of his own further experiments; and, in fact, with far better opportunities than fell to the lot of Mr. Robins, we have shown that inflamed gunpowder is about double the strength that he has assigned to it, and that it expands itself with the velocity of about 5000 feet per second.

4. Fully sensible of the importance of experiments of this kind, first practised by Mr. Robins with musket balls only, my endeavours for many years were directed to the prosecution of the same, on a larger scale, with cannon balls; and having had the honour to be called on to give my assistance at several courses of such experiments, carried on at Woolwich by the ingenious officers of the Royal Artillery there, under the auspices of the Masters General of the Ordnance, I have assiduously attended them for many years. The first of these courses was performed in the year 1775, being 2 years after my establishment in the Royal Academy at that place: and in the Philos. Trans. for the year 1778, I gave an account of these experiments, with deductions, in a memoir, which was honoured with the Royal Society's gold medal of that year, as before mentioned in the beginning of the 34th Tract, in the last volume; where also are stated the results of the first experiments, as well as those of the second course of experiments, performed in the years 1783, 1784, 1785, 1786.

5. An ample account is given of these experiments, and the results deduced from them in the 34th Tract; some few circumstances only of which may be repeated here. In this course, four brass guns were employed, very nicely bored and cast on purpose, of different lengths, but equal in all other respects, viz, in weight and bore, &c. The lengths of the bores of the guns were,

the gun n° 1, was 15 calibres, length of bore 28·5 inc.
n° 2, . 20 calibres, ... 38·4
n° 3, . 30 calibres, ... 57·7
n° 4, . 40 calibres, ... 80·2

the calibre of each being $2\frac{1}{50}$ inches, and the medium weight of the balls 16 oz 13 drams.

6. The mediums of all the experimented velocities of the balls, with which they struck the pendulous block of wood, placed at the distance of 32 feet from the muzzle of the gun, for several charges of powder, were as in the following table,

Table of Initial Velocities.

Powder.	The Guns.			
oz	N° 1.	N° 2.	N° 3.	N° 4.
2	780	835	920	970
4	1100	1180	1300	1370
6	1340	1445	1590	1680
8	1430	1580	1790	1940
12	1436	1640	.	.
14	.	1660	.	.
16	.	.	2000	.
18	.	.	.	2200

placed in the 1st column, for all the four guns, the numbers denoting so many feet per second. Whence in general it appears how the velocities increase with the charges of powder, for each gun, and also how they increase as the guns are longer, with the same charge, in every instance.

7. By increasing the quantity of the charges continually, for each gun, it was found that the velocities continued to increase till they arrived at a certain degree, different in each

gun; after which, they constantly decreased again, till the bore was quite filled with the charge. The quantities of powder, when the velocities arrived at their maximum or greatest state, were various, as might be expected, according to the lengths of the guns: the weight of powder, with the length it extended in the bore, and the fractional part of the bore it occupied, are shown in the following table, of the charges for the greatest effect.

Gun, n°	Length of the bore.	The Charge.		
		Weight,	Length.	
		oz	Inches	Part of whole
1	28·5	12	8·2	$\frac{3}{10}$
2	38·4	14	9·5	$\frac{3}{12}$
3	57·7	16	10·7	$\frac{3}{16}$
4	80·2	18	12·1	$\frac{3}{20}$

8. Some few experiments in this course were made to obtain the ranges and times of flight, the mediums of which are exhibited in the following table.

Guns.	Powder	Ball's			Eleva. gun.	Time of flight.	Range	First veloc.
		wt		diam				
	oz	oz	dr	inches		sec	feet	feet
n° 2	2	16	10	1·96	45°	21·2	5109	863
do.	2	16	5	1·96	15	9·2	4130	868
do.	4	16	8	1·96	15	9·2	4660	1234
do.	8	16	12	1·96	15	14·4	6066	1644
do.	12	16	12	1·95	15	15·5	6700	1676
n° 3	8	15	8	1·96	15	10·1	5610	1938

In this table are contained the following concomitant data, determined with a tolerable degree of precision; viz, the weight of the powder, the weight and diameter of the ball, the initial or projectile velocity, the angle of elevation of the gun, the time in seconds of the ball's flight through the air, and its range, or the distance where it fell on the horizontal

plane. From which it is hoped that some aid may be derived towards ascertaining the resistance of the medium, and its effects on other elevations, &c, and so afford some means of obtaining easy rules for the cases of practical gunnery.

9. Another subject of enquiry in the foregoing experiments, was, how far the balls would penetrate into solid blocks of elm wood, fired in the direction of the fibres. The annexed tablet shows the results of a few of the trials that were made with the gun n° 2, with the most frequent charges of 2, 4, and 8 ounces of powder; and the mediums of the penetrations, as placed in the last line, are found to be 7, 15, and 20 inches, with those charges. These penetrations are nearly as the numbers

Penetrations of Balls into solid Elm wood.

Powder 2	4	8 oz
7	16·6	18·9
	13·5	21·2
		18·1
		20·8
		20·5
Means 7	15	20

2, 4, 6, or 1, 2, 3; but the charges of powder are as 2, 4, 8, or 1, 2, 4; so that the penetrations are proportional to the charges as far as to 4 ounces, but in a less ratio at 8 ounces; whereas, by the theory of penetrations, the depths ought to be proportional to the charges throughout, or, which is the same thing, as the squares of the velocities. So that it seems the resisting force of the wood is not uniformly or constantly the same, but that it increases a little with the increased velocity of the ball. This may probably be occasioned by the greater quantity of fibres driven before the ball; which may thus increase the spring and resistance of the wood, and prevent the ball from penetrating so deep as it otherwise might do.

10. From a general inspection of this second course of these experiments it appears, that all the deductions and observations made on the former course, are here corroborated and strengthened, respecting the velocities and weights of the balls, and charges of powder, &c. It further appears also,

that the velocity of the ball increases with the increase of charge only to a certain point, which is peculiar to each gun, where it is greatest; and that by further increasing the charge, the velocity gradually diminishes, till the bore is quite full of powder. That this charge for the greatest velocity is greater as the gun is longer, but yet not greater in so high a proportion as the length of the gun is; so that the part of the bore filled with powder, bears a less proportion to the whole bore in the long guns, than it does in the shorter ones; the part which is filled being indeed nearly in the inverse ratio of the square root of the empty part.

11. It appears that the velocity, with equal charges, always increases as the gun is longer; though the increase in velocity is but very small in comparison to the increase in length; the velocities being in a ratio somewhat less than that of the square roots of the length of the bore, but greater than that of the cube roots of the same, and is indeed nearly in the middle ratio between the two.

12. It appears, from the table of ranges, that the range increases in a much lower ratio than the velocity, the gun and elevation being the same. And when this is compared with the proportion of the velocity and length of gun in the last paragraph, it is evident that we gain extremely little in the range by a great increase in the length of the gun, with the same charge of powder. In fact, the range is nearly as the 5th root of the length of the bore; which is so small an increase, as to amount only to about a 7th part more range for a double length of gun.—From the same table it also appears, that the time of the ball's flight is nearly as the range; the gun and elevation being the same.

13. It has been found, by these experiments, that no difference is caused in the velocity, or range, by varying the weight of the gun, nor by the use of wads, nor by different degrees of ramming, nor by firing the charge of powder in different parts of it. But that a very great difference in the velocity arises from a small degree in the windage: indeed with the usual established windage only, viz, about $\frac{1}{20}$ of the

calibre, no less than between $\frac{1}{3}$ and $\frac{1}{4}$ of the powder escapes and is lost: and as the balls are often smaller than the regulated size, it frequently happens that half the powder is lost by unnecessary windage.

14. It appears too that the resisting force of wood, to balls fired into it, is not constant: and that the depths penetrated by balls, with different velocities or charges, are nearly as the logarithms of the charges, instead of being as the charges themselves, or, which is the same thing, as the square of the velocity.—Lastly, these and most other experiments, show, that balls are greatly deflected from the direction in which they are projected: and that as much as 300 or 400 yards in a range of a mile, or almost $\frac{1}{4}$th of the range.

15. Proceed we now to state the deductions and applications of the experiments made in the years 1787, 1788, 1789, 1791. These experiments, we have seen, were chiefly instituted to obtain the effects of the air's resistance to balls, in their rapid flight through it. To determine the resistance to the very high velocities, were employed balls of three several sizes, viz. of 2 inches, 2·78 inches, and 3·55 inches in diameter. These were discharged with various degrees of velocity, from 300 feet to 2000 feet in a second of time; and they were also made to strike the pendulum block at several different distances from the guns, in order to obtain the quantity of velocity lost, in passing through those spaces of air; whence the degrees of resistance were obtained, appropriate to the different velocities. These series of resistances, for the three sizes of balls above-mentioned, have been obtained in a state remarkably regular, not only each series in itself, but also in comparison with each other; the terms in every one of them following a certain uniform law, in respect of the velocity, being indeed nearly as the $2\frac{1}{10}$ power of the velocity; and the terms of any one series also, as compared with the corresponding terms of another, with the same velocity, these being in a constant proportion to one another, viz, as the surfaces of the balls moved nearly, or as the squares of their diameters, with about $\frac{1}{20}$ part more

in counting from the less ball to the greater, or $\frac{1}{24}$ part less when comparing the greater ball to the less.

16. The same laws of resistance were also found to obtain in the slower motions, with the whirling-machine in the 36th Tract, both in respect of the different velocities with the same body, and of the different bodies with the same velocity. From which uniformity of effects it happens, that the numbers resulting from the larger velocities, in the one course of experiments, and those derived from the slow motions in the other course, form as it were the terms, in the different parts of one and the same general series of resistances. These circumstances enable us, easily and justly, to interpolate and fill up the intermediate places, between 300 feet, the smallest of the rapid motions, and 20 feet, the greatest of the slow ones. And hence we obtain the following complete series, from the smallest to the greatest, for a ball of just 2 inches in diameter.

17. *Table of the Air's Resistance to a Ball of 2 inches Diameter.*

Veloc. per sec. in feet.	Resist. by exper.	Resist. by theory.	Ratio of expos. to theory.	Resist. as the powers of the veloc.	Resistances in lbs.
feet	oz	oz			
5	0·006	0·005	1·20		
10	0·026	0·021	1·23		
15	0·058	0·046	1·25		
20	0·103	0·082	1·26		
25	0·163	0·128	1·27		
30	0·237	0·184	1·29		
40	0·427	0·329	1·30		
50	0·676	0·511	1·32		
100	2·78	2·046	1·36		0·174
200	11·34	8·18	1·39	2·028	0·709
300	25·3	18·4	1·42	2·028	1·612
400	46·5	32·7	1·43	2·032	2·906
500	74·4	51·2	1·46	2·042	4·650
600	110·4	73·6	1·50	2·055	6·900
700	156·0	100·2	1·55	2·069	9·750
800	212·0	130·9	1·62	2·084	13·250
900	280·3	165·7	1·69	2·099	17·519
1000	362·1	204·6	1·77	2·115	22·625
1100	456·9	247·6	1·84	2·128	28·556
1200	564·4	294·6	1·91	2·138	35·275
1300	683·3	345·7	1·98	2·146	42·706
1400	811·5	401·0	2·03	2·151	50·719
1500	947·1	460·1	2·06	2·153	59·194
1600	1086·9	523·7	2·08	2·152	67·931
1700	1228·4	591·2	2·07	2·150	76·775
1800	1368·6	662·8	2·06	2·145	85·537
1900	1505·7	738·5	2·03	2·141	94·106
2000	1637·8	818·3	2·00	2·136	102·362

18. In the first column of this table are contained the series of velocities, to the ball of 2 inches diameter, from that of 5 feet up to 2000 feet per second of time; and in the 2nd column are placed the correspondent resistances, expressed in avoirdupois ounces and decimals. For the purposes of comparison, in the 3rd column are placed the cor-

respondent degrees of resistance to the same ball, as computed from the known theory of such resistances, viz, considered as being proportional to the square of the velocity. The theorem for this purpose is investigated in prob. 20, pa. 367, vol. 2 of my Course of Mathematics, and is this, $\frac{pnd^2}{32g}v^2 = r$ the resistance, where p is $= 3{\cdot}1416$, $n =$ the specific gravity of the medium, $g = 16$, $d =$ the ball's diameter, and $v =$ its velocity, both expressed in feet. Now, for our ball of 2 inches or $\frac{1}{6}$ of a foot in diameter, that is, $d = \frac{1}{6}$, and the value of $n = 1\frac{1}{3}$ ounces, the mean specific gravity of air, the theorem becomes $\frac{1}{4888}v^2$ or $\frac{9}{44000}v^2 = r$, the resistance. Hence all the series of resistances will be easily found, by dividing the squares of the velocities by the constant number 4888; or, still easier, by dividing the squares of the triple velocities by 44000. In this manner then were easily found the theoretical resistances, as placed in the 3rd column of the table.

19. Then, to form a comparison between these two columns, of the experimental and theoretical resistances, each number in the former is divided by the correspondent number in the latter, and the ratios or quotients are set opposite them in the next or 4th column; where it is seen that, near the beginning, or with the slowest motions, the experimental or real resistance, exceeds the theoretical one, by about the 5th part, or in the ratio of 6 to 5; that as the velocity is increased, the former gains more and more over the latter, all along, till at the velocity of 1600 feet per second, where the difference is at the greatest, the former being more than double the latter, or in the ratio of 2·08 to 1; after which, as the velocity is further increased, the ratio slowly decreases again, and terminates at 2000 feet velocity, in the ratio of 2 to 1 very nearly.

20. To the former 4 columns a 5th is added, to show according to what power of the velocity, at every point, the resistance increases, being the indices of those powers, by comparing constantly a first number, for instance that of

the 100 feet velocity, with each of the following ones separately, first the 100 with the 200, then the 100 with the 300, then the first with the 400, and so on to the end; by which means it appears that, commencing with the 2nd power or square of the velocity, at the very beginning or slowest motion, the exponent of the power gradually increases, till, at the velocity of 1500 or 1600 feet, it arrives at the 2·153 power of the same, being nearly the $2\frac{1}{7}$ power, or $2\frac{2}{13}$ more nearly, that is, the resistance to the 100 feet velocity, is to that of the 1600 feet velocity, as $100^{2\cdot153}$ to $1600^{2\cdot153}$, or as $1^{2\cdot153}$ to $16^{2\cdot153}$, or as 1 to $16^{2\cdot153}$, since all the powers of 1 are equal to 1 only; so that, if r be put to denote 2·78 ounces, the resistance to the velocity of 100 feet velocity, then $2\cdot78 \times 16^{2\cdot153}$ will be the resistance to the 1600 feet velocity; and so on for the others. After the 1600 feet velocity, where the exponent (2·153) is greatest, it gradually decreases again to the end. It may further be noticed, that the arithmetical medium among all these exponents of the powers, is 2·104, or $2\frac{1}{10}$ nearly. The 6th or last column is added, to show the resistances in avoirdupois pounds.

21. The method of determining these indices of the powers, is the same as employed in art. 43 of the 36th Tract; viz, having given any two velocities v, v', and their correspondent resistances r, r'; to find, according to what powers of v and v', the resistances r, r' are proportional. Let x denote the exponent of the power of the two velocities v, v'; then $v^x : v'^x :: r : r'$; hence, dividing the consequents by the antecedents, gives $1 : (\frac{v'}{v})^x :: 1 : \frac{r'}{r}$; therefore $(\frac{v'}{v})^x = \frac{r'}{r}$; taking the logarithms gives $x \times \log. \frac{v'}{v} = \log. \frac{r'}{r}$; hence $x = \frac{\log. r' - \log. r}{\log. v' - \log. v}$ that is, the exponent of the power of the velocities, to which the resistances are proportional, is equal to the quotient of the differences of the logs. of the resistances, divided by the difference of the logs. of the velocities. Now, to apply this theorem to the calculation from the numbers in the preceding table of our experiments, beginning at the velocity of 100, and thence to find the exponents for all the

succeeding velocities 200, 300, 400, &c, the continual quotients $\frac{v'}{v}$, of each of these velocities 200, 300, 400, &c, divided by the first 100, give the successive numbers 2, 3, 4, &c, the logs. of which therefore are the correspondent divisors to the difference between the logs. of the first resistance and that of each of the following ones, to give the respective numbers denoting the indices of the powers, as arranged in the 5th column of the table.

22. The circumstance of the variable and increasing exponent in the ratio of the resistance, is owing chiefly to the increasing degree of vacuity left behind the ball, in its flight through the air, and to the condensation of the air before it. It is well known, that air can only rush into a vacuum with a certain degree of velocity, viz, about 1200 or 1400 feet in a second of time; therefore, as the ball moves through the air, there is always left behind a kind of vacuum, either partial or complete; that as the velocity is greater, the degree of vacuity behind goes on increasing, till at length, when the ball moves as rapidly as the air can rush in and follow it, the vacuum behind the ball is complete, and so continues complete ever after, as the ball continues to move with all greater degrees of velocity. Now the resistance, which the ball suffers in its flight, is of a triple nature; one part of it being in consequence of the *vis inertia* of the particles of air, which the ball strikes in its course; another part from the accumulation of the elastic air before the ball; and the third part arises from the continued pressure of the air on the forepart of the ball, when the velocity of this is such as to leave a vacuum behind it in its flight, either wholly or in part: for, while the ball is at rest, it is manifest that this pressure of the whole atmosphere is the same or equal on all sides of the ball; but, as soon as the ball begins to move, it is also manifest that the pressure behind will be less than the constant degree of pressure in front, and the difference must be the greater as the motion of the ball is the more rapid, being in fact proportional nearly as the velocity of the ball,

as compared with that of air rushing into a complete vacuum, that is, while the former is not greater than the latter; for, as soon as the motion of the ball becomes equal to that of the air, and always when greater, then the ball has to sustain the whole pressure of the atmosphere on its forepart, without having any aid from a counter-pressure behind.

23. Thus then we see that the resistance against the ball is two-fold, (besides that arising from the unknown degree of compression before the ball), the one arising from percussion, by the ball striking and displacing the particles of air in its path, and which increases continually in the duplicate proportion of the ball's velocity; and the other from the weight of the atmosphere, increasing with that velocity, to which, being of the nature of pressure, it is proportional; but arriving at its maximum when that is equal to or exceeds the velocity of air into a vacuum, after which it is a constant quantity for all greater degrees of velocity. These circumstances then very well show the reason why the experimented resistance proceeds in a ratio increasing gradually more and more above the square of the velocity, till this exceeds 12 or 14 hundred feet, the motion of air into a vacuum, and then rather decreases again. So that it appears that the whole estimatable resistance consists of two parts; of which the one part is proportional to the square of the velocity, and the other is simply as the velocity only.

24. Besides the experiments hitherto noticed, others were at several times performed, both with guns and mortars, discharged with different degrees of elevation and velocities, to observe the extent of the ranges, the times of flight, and the angles at which the shots fall to the ground; a synopsis of such experiments of this kind as have been made, is contained in the following table.

N°	Eleva. of the piece.	The Ball's wt	The Ball's diam	The Ball's velocity	Time of flight	Range	Remarks.
	degr.	lbs	inches	feet	sec	feet	
1	45°	1·04	1·96	863	21″·2	5109	
2	15	1·02	1·96	868	9·2	4130	
3	15	1·03	1·96	1234	9·2	4660	One pound balls.
4	15	1·06	1·96	1644	14·4	6066	
5	15	1·06	1·96	1676	15·5	6700	
6	15	1·03	1·96	1630	10·1	5610	
7	45	14·25	4·43	197	4·5	256	Shells filled with lead.
8	45	14·25	4·43	297	7·0	870	
9	45	12·88	4·49	207	5·0	376	Solid iron balls
10	45	12·88	4·49	293	8·0	1068	
11	45	8·70	4·43	252	5·0	368	Shells filled with water.
12	45	8·56	4·43	380	9·0	1083	
13	45	8·35	4·44	257	5·7	426	Empty shell.
14	45	1·38	4·40	633	6·0	505	Oaken ball.
15	45	0·40	4·40	1175	5·5	408	Paper ball.
16	60	64·0	7·69	564	24·5	4350	Solid ball.
17	60	48·0	7·69	651	26·8	4950	Shell.
18	60	128·0	9·74	556	25·5	5220	Solid ball.
19	60	96·0	9·74	643	27·0	5790	Shell.
20	45	3·0	2·78	1200		7360	.
21	45	3·0	2·78	860	23·0	5850	.
22	45	3·0	2·78	600	16·0	3870	Balls
23	45	3·0	2·78	860	22·5	5850	
24	30	3·0	2·78	1200	22·0	7680	
25	20	3·0	2·78	1200	12·0	4300	.
26	45	3·0	2·78	1200	ball	7360	.
27	45	3·0	2·78	1697	d°	8130	Ang. of fall 61°¼
28	75	3·0	2·78	1200	d°	5960	
29	75	3·0	2·78	1697	d°	6130	
30	35	48·0	7·69	571	shell	6150	Angle of fall 52°
31	55	48·0	7·69	571	d°	5100	61¼
32	45	96·0	9·74	430	d°	6080	

Some applications of the foregoing principles we now proceed to state, in the following problems.

PROBLEM I.

25. *To determine the Resistance of the Medium against a Ball of any other size, moving with any of the Velocities contained in the foregoing table.*

The experiments, that have been so amply detailed, sufficiently prove that the resistances increase in a rather higher ratio than the surfaces of different bodies, with the same velocity. Yet the ratios of the resistances, and of the surfaces, or of the squares of the diameters, which is the same thing, are so nearly alike, that they may generally be considered as equal to each other, in most calculations relating to artillery, or other practice. For example, to compare some of the experiments which we have made with the three different sizes of balls, viz, of 2 inches, and 2·78 inches, and 3·55 inches diameter; of which the table of resistances for the first was given in art. 17 preceding, and also in art. 145 of tract 34, as far as to 2000 feet velocity; that for the 2nd size in art. 167 of that tract, from 900 to 1700 feet velocity; and that for the 3rd size, in art. 193 of the same tract, from 1200 to 1800 feet velocity. Now the three diameters being 2, and 2·78, and 3·55, the squares of which are proportional to the three numbers 1, and 1·931, and 3·15, which are also the ratios of the three surfaces. Now taking any one of the velocities that is common to all the three tables, as suppose that of 1200 feet per second, we find the three resistances, corresponding to this velocity, are 564·5, and 1133, and 1840, which three numbers are in the ratios of 1, and 2·01, and 3·26, where the two latter numbers exceed the corresponding two of the former, the one by the 25th part, and the other by the 35th part. Again, let us compare the resistances for the 1700 feet velocity, which are 1228·4, and 2472, and 4010, which we find are in the ratios of 1, and 2·013, and 3·26, which are again nearly in the same ratios as the former. So that it appears, that the resistances in general may be considered as increasing above the ratio of the surfaces, by about the 30th part only. For example, sup-

pose it were required to determine what would be the resistance of the air against a 24lb ball discharged with a velocity of 2000 feet per second of time. Now, by the 1st of the foregoing tables, the ball of 2 inches diameter, when moving with the velocity 2000, suffered a resistance of 1638 ounces, or 102lb; then, since the resistances, with the same velocity, are as the surfaces nearly; and the surfaces are as the squares of the diameters; and the diameters being 2 and 5·6 nearly, the squares of which are 4 and 31·36, therefore as 4 : 31·36 : : 102lb : 800lb nearly, that is, the 24lb ball would suffer the enormous resistance of 800lb in its flight, in opposition to the direction of its motion!

And, in general, if the diameter of any proposed ball be denoted by d, and r denote the resistance in the last table due to the proposed velocity of the 2 inch ball; then $\frac{1}{4}d^2r$ will denote the resistance with the same velocity against the ball whose diameter is d.

PROBLEM II.

26. *To assign a Rule for determining the Resistance due to any Indeterminate Velocity of a Given Ball.*

This problem is very difficult to be performed near the truth, on account of the variable ratio which the resistance bears to the velocity, increasing always more and more above that of the square of the velocity, at least to a certain extent; and indeed it appears that there is no single integral power whatever of the velocity, or no expression of the velocity in one term only, that can be proportional to the resistances throughout. It is true indeed, that such an expression can be assigned by means of a fractional power of the velocity, or rather one whose index is a mixed number, viz, $2\frac{1}{10}$ or 2·1; thus $\frac{v^{2\cdot1}}{5700}$ = the resistance, is a formula in one term only, which will answer to all the numbers in the last table of resistances very nearly, for the ball of 2 inches diameter; and consequently, by means of the ratio of the squares of the diameters of the balls, for any other balls whatever. But this

formula, though serving very well for some particular resistance, or even for constructing a complete series or table of resistances, is not proper for the use of problems in which fluxions and fluents are concerned, on account of the mixed number $2\frac{1}{10}$, in the index of the velocity v.

We must therefore have recourse to an expression in two terms, or a formula containing two integral powers of the velocity, as v^2 and v, the first and second powers, affected with general coefficients m and n, as $mv^2 + nv = r$, the resistance. Now, to determine the general numerical values of the coefficients m and n, we must adapt this general expression $mv^2 + nv = r$, to two particular cases of velocity, at a convenient distance from each other, in one of the foregoing tables of resistances, as the last for instance. Now, after making several trials in this way, it is found that the two velocities of 500 and 1000 answer the general purpose better than any other that has been tried. Thus then, employing these two cases, we must first make $v = 500$, and $r = 74{\cdot}4$ oz $= 4{\cdot}65$lb, its correspondent resistance, and then again $v = 1000$, and $r = 362$ oz $= 22\frac{5}{8}$lb, the resistance belonging to it: this will give two equations, by which the general value of m and of n will be determined. Thus then the two equations being

$$500^2m + 500n = 4{\cdot}65,$$
$$\text{and } 1000^2m + 1000n = 22{\cdot}625;$$

dividing the 1st by 500, and the 2d by 1000, they are $\begin{cases} 500m + n = {\cdot}0093, \\ 1000m + n = {\cdot}022625; \end{cases}$

the dif. of these is . . . $500m = {\cdot}013325$,

and therefore div. by 500, gives $m = {\cdot}00002665$;

hence $n = {\cdot}0093 - 500m = {\cdot}0093 - {\cdot}013325 = -{\cdot}00665 = n$.

Hence then the general formula will be ${\cdot}00002665v^2 - {\cdot}004025v = r$, the resistance nearly in avoirdupois pounds, in all cases or all velocities whatever.

27. Now, to find how near to the truth this theorem comes, in every instance in the table, by substituting for v, in this formula, all the several velocities, 100, 200, 300, &c, to 2000, these give the correspondent values of r, or the resistances, as in the 2d column of the annexed table, their velocities being in the first column; and the real experimented resistances are set opposite to them in the 3d or last column of the same. By the comparison of the numbers in these two columns together, it is seen that there are no where any great difference between them, being sometimes a little in excess, and again a little in defect, by very small differences; so that, on the whole, they will nearly balance one another, in any particular instance of the range or flight of a ball, in all degrees of its velocity, from the first or smallest, to the last or greatest. Except in the first two or three numbers, at the beginning of the table, for the velocities 100, 200, 300, for which cases another theorem may be employed. Now, in these three velocities, as well as in all that are smaller, down to nothing, the theorem $\cdot 0000176v^2 = r$ the resistance, will very well serve, as it brings out for the first three resistances ·176, and ·704, and 1·584, differing in those cases by a very small fraction only.

Velocs. or v.	Comput. resists.	Exper. resists.
100	—·133	·174
200	+·266	·709
300	1·200	1·612
400	2·666	2·906
500	4·666	4·650
600	7·200	6·900
700	10·206	9·750
800	13·866	13·250
900	18·000	17·519
1000	22·666	22·625
1100	27·866	28·556
1200	33·266	35·275
1300	39·866	42·706
1400	46·666	50·719
1500	54·000	59·194
1600	61·866	67·931
1700	70·466	76·775
1800	79·200	85·537
1900	88·666	94·106
2000	98·666	102·362

The same otherwise.

28. If however the foregoing rule, $(\cdot 00002665v - \cdot 004025)\, v = r$, for the ball of 2 inches diameter, be not in all cases quite near enough, as it varies from the truth so much as $\frac{1}{11}$

part nearly in one place, viz, when the velocity is about 1600 or 1700 feet per second; we may find a more exact formula in the following manner. Let us suppose the resistance to be proportional to three different powers of the velocity, that is, partly as the square v^2, and partly as the first power v, and partly as the $\frac{3}{2}$ power $v^{\frac{3}{2}}$, with indeterminate coefficients, or expressed by $xv^2 + yv^{\frac{3}{2}} + zv = r$, or $(xv + y\sqrt{v} + z)v = r$; then the values of the three assumed general quantities x, y, z, will be determined, by assuming r equal to the true or experimented resistance at three several parts of the series, as at the three velocities 600, 1200, 1800 feet, by which means the resulting formula will be quite correct for all these three velocities, and very nearly so for all the other or intermediate ones. Now, the experimented resistances for those three velocities are 6·9, and 35·275, and 85·537; these numbers then being substituted severally for r, there results the three following equations,

$$600^2x + 600^{\frac{3}{2}}y + 600z = 6{\cdot}9,$$
$$1200^2x + 1200^{\frac{3}{2}}y + 1200z = 35{\cdot}275,$$
$$1800^2x + 1800^{\frac{3}{2}}y + 1800z = 85{\cdot}537.$$

Then, dividing the first of these by 600, the second by 1200, and the third by 1800, there result the three

$$600x + y\sqrt{600} + z = {\cdot}0115,$$
$$1200x + y\sqrt{1200} + z = {\cdot}029396,$$
$$1800x + y\sqrt{1800} + z = {\cdot}0475205.$$

Now, subtracting the first of these from the 2nd, and the 2nd from the 3rd; there result these two,

$$600x + (\sqrt{1200} - \sqrt{600})y = {\cdot}017896,$$
$$600x + (\sqrt{1800} - \sqrt{1200})y = {\cdot}0181245.$$

And the difference of these two is

$(\sqrt{1800} - 2\sqrt{1200} + \sqrt{600})y = {\cdot}0002285$, or div. by $\sqrt{600}$ it is $(\sqrt{3} - 2\sqrt{2} + \sqrt{1})y = {\cdot}00000933$, and then

$$y = \frac{{\cdot}00000933}{\sqrt{3} - \sqrt{8} + 1} = \frac{{\cdot}00000933}{-{\cdot}096376} = -{\cdot}0000968, \text{ the value of } y.$$

Hence $600x = {\cdot}017896 - (\sqrt{1200} - \sqrt{600})y = {\cdot}018878$, and also $600x = {\cdot}0181245 - (\sqrt{1800} - \sqrt{1200})y = {\cdot}018878$, therefore $x = \frac{{\cdot}018878}{600} = {\cdot}000031463$, the value of x.

Then $z = \cdot 0115 - 600x - y\sqrt{600} = -\cdot 005007$,
or $z = \cdot 029396 - 1200x - y\sqrt{1200} = -\cdot 005007$,
and $z = \cdot 0475205 - 1800x - y\sqrt{1800} = -\cdot 005007$,
all the three values the same, showing so far the truth of these deductions. Hence the general formula for the resistance will be nearly

$$(\cdot 00003146v - \cdot 0000968\sqrt{v} - \cdot 005)\, v = r,$$
$$\text{or } (\cdot 0000315v - \cdot 0001\sqrt{v} - \cdot 005)\, v = r,$$

which is quite correct for the three velocities, 600, 1200, 1800, and very nearly so for all the others.

29. But, further, the middle turn $\cdot 0001\sqrt{v}$ may be expunged, and so the formula rendered much more simple, in the following manner. By adapting this formula to a few of the velocities, in different parts of the series, it is found, not only that the 2nd and 3rd terms, $\cdot 0001\sqrt{v}$ and $\cdot 005$ are always both very small in respect of the first term $\cdot 00003146v$, but also that the 2nd term, $\cdot 0001\sqrt{r}$, may be generally taken at about $\frac{3}{5}$ of the 3rd term $\cdot 005$, or nearly $= \cdot 003$ on a medium; therefore these two small terms together are nearly $= \cdot 008$; consequently the foregoing formula becomes $(\cdot 00003146v - \cdot 008)\, v = r$ nearly.

30. Computing now, by this formula, all the series of the resistances, or values of r, for all the several velocities, it is found that they come out a little nearer than those by the former theorem $(\cdot 00002665v - \cdot 004)\, v = r$, but having differences from the true resistances the reverse way, most commonly, the one from the other, that is, the one being in excess, for the most part, when the other is in defect. This circumstance then naturally suggests the idea of taking the medium between the two, for a still nearer formula, that is, half the sum

$$\text{of } (\cdot 00002665v - \cdot 004)v = r,$$
$$\text{and } (\cdot 00003146v - \cdot 008)v = r,$$
$$\text{which is } (\cdot 00002906v - \cdot 006)v = r,$$
$$\text{or nearly } (\cdot 0000291v - \cdot 006)v = r,$$

the diameter of the ball being 2 inches; and therefore, for any other diameter d, the resistance will be

$$(\cdot 0000073\, v - \cdot 0015)vd^2 = r.$$

31. Now, by computing, by the preceding formulæ, the resistances for all the velocities, they come out as in the following table.

Velocity	True resistance by exper.	Resistance by the Formulæ.				
		$\cdot 00002665v^2 - 004v$.	$\cdot 00003146v^2 - \cdot 008v$.	$\cdot 00002906v^2 - \cdot 006v$	$\cdot 00003026v^2 - \cdot 007v$	$\cdot 00003028v^2 - \cdot 007\frac{1}{2}v + 0\cdot 3$
feet	lbs.	lbs	lbs.	lbs.	lbs	
100	·17	−·13	−·48	−·31	−·39	−0·11
200	·71	+·27	−·34	−·04	−·19	+0·08
300	1·61	1·20	+·43	+·82	+·63	0·88
400	2·91	2·67	1·83	2·26	2·05	2·29
500	4·65	4·67	3·87	4·28	4·08	4·29
600	6·90	7·20	6·53	6·87	6·70	6·90
700	9·75	10·21	9·82	10·06	9·94	10·12
800	13·25	13·82	13·73	13·82	13·77	13·95
900	17·52	18·00	18·28	18·17	18·22	18·38
1000	22·63	22·67	23·46	23·11	23·28	23·41
1100	28·56	27·87	29·27	28·61	28·94	29·06
1200	35·28	33·27	35·70	34·70	35·20	35·30
1300	42·71	39·87	42·77	41·38	42·08	42·15
1400	50·72	46·67	50·46	48·64	49·55	49·62
1500	59·19	54·00	58·79	56·48	57·64	57·69
1600	67·93	61·87	67·74	64·90	66·32	66·36
1700	76·78	70·47	77·32	73·90	75·61	75.63
1800	85·54	79·20	87·53	83·48	85·50	85·52
1900	94·11	88·67	98·37	93·65	96·01	95·99
2000	102·36	98·67	109·85	104·00	106·92	107·09
1	2	3	4	5	6	7

Where the first column contains the series of velocities, the 2nd column the series of true or experimented resistances; the 3rd the resistances as computed by the formula $\cdot 00002665v^2 - \cdot 004v$, or $(\cdot 02665v - 4) \times \cdot 001v$, in which we perceive that the numbers agree nearly with the true resistances as far as to the velocity 1100, after which they are gradually more and more in defect, till the error amounts to more than 7 out of 77, or about the $\frac{1}{11}$ part, and which therefore deviates too much from the truth in the higher velocities. Again, the 4th column contains the resistances as com-

puted from the formula $\cdot 00003146v^2 - \cdot 008v$ or $(\cdot 03146v - 8) \times \cdot 001v$, which agree nearly with the true resistances as far as to the velocity 1700, after which they are in excess, having in the 1800 velocity 2 in excess, or the 43rd part; in the 1900 velocity, 4 in excess, or the 24th part; and at the 2000 velocity, 7 in excess, or the 15th part. Further, the 5th column contains the resistances as computed from the theorem $\cdot 00002906v^2 - \cdot 006v$, or $(\cdot 02906v - 6) \times \cdot 001v$, being the medium between the two preceding ones; and these numbers are nearer the truth than either of the former, the greatest error, in defect, being, at 1700 velocity, between 3 and 4, or about the 20th part. Also, in the 6th column are exhibited the resistances as computed by the theorem $\cdot 00003026v^2 - \cdot 007v$ or $(\cdot 03026v - 7) \times \cdot 001v$, being the medium between the two preceding it, which may be esteemed sufficiently near the truth, varying but by two or three units in the last two velocities.

But, it must still be understood that for the first three or four velocities, the foregoing simple rule $\cdot 0000176v^2$, will always give the nearest value of the true resistance.

32. Lastly, another theorem may be found still a little nearer the experiments, on this principle, viz, in three terms, containing the square of the velocity, and the velocity, and a constant quantity, as $xv^2 + yv + z$. Now, expounding v severally by the three velocities 600, 1200, and 1800, and making the results equal to their correspondent resistances, we obtain the following near values of the coefficients, viz, $x = \cdot 00003028$, and $y = - \cdot 00716$, and $z = 0 \cdot 3$; by which the assumed theorem becomes $\cdot 00003028v^2 - \cdot 007\frac{1}{6}v + 0 \cdot 3$, nearly. Then the resistances computed by this formula, for all the velocities, or values of v, they are as arranged in the 7th or last column, and are mostly nearer than the preceding ones, being only a littte too small in the first two or three numbers, and too large in the last two numbers. Indeed most of these theorems exhibit the last two numbers as larger than the experimented ones; whence it may be suspected,

that these two numbers are really too small, and that they should probably be about 95 and 104.

33. *Corol.* 1. The foregoing rule $\cdot 00003026v^2 - \cdot 007v = r$, in the 6th column, denotes the resistance for the ball in the table whose diameter is 2, the square of which is 4; hence to adapt it to a ball of any other diameter d, we have only to alter the former in proportion to the squares of the diameters, by which it becomes $\frac{1}{4}(\cdot 00003026v^2 - \cdot 007v)d^2 = (\cdot 000007565v^2 - \cdot 00175v)d^2$, which is the resistance for the ball whose diameter is d, with the velocity v.

34. *Corol.* 2. And, in a similar manner, to adapt the theorem $\cdot 0000176v^2 = r$, for the smaller velocities, to any other size of ball, we must multiply it by $\frac{1}{4}d^2$, the ratio of the surfaces, by which it becomes $\cdot 0000044d^2v^2 = r$.

We shall soon take occasion to make some applications in the use of the foregoing formulas, after considering the effects of such velocities in the case of nonresistance.

PROBLEM III.

35. *To determine the Height to which a Ball will rise, when discharged from a Cannon Perpendicularly Upwards with a Given Velocity, in Nonresisting Space, or supposing no Resistance in the Air.*

By art. 73 pa. 151 vol. 2 of the Academy Course, it appears that any body projected upwards, with a given velocity, will ascend to the height due to the velocity, or the height from which it must naturally fall to acquire that velocity; and the spaces fallen being as the square of the velocities; also 16 feet being the space due to the velocity 32; therefore the space due to any proposed velocity v, will be found thus, as $32^2 : 16 :: v^2 : s$ the space, or as $64 : 1 :: v^2 : \frac{1}{64}v^2 = s$ the space, or the height to which the velocity v will cause the body to rise, independent of the air's resistance.

Exam. For example, if the first or projectile velocity, be 2000 feet per second, being nearly the greatest experimented

velocity; then the rule $\frac{1}{64}v^2 = s$ becomes $\frac{1}{64} \times 2000^2 =$ 62500 feet $= 11\frac{5}{6}$ miles; that is, any body, projected with the velocity 2000 feet, would ascend nearly 12 miles in height, without resistance.

36. *Corol.* Because, by art. 88 Projectiles vol. 2 of the Course, the greatest range is just double the height due to the projectile velocity, therefore the range, at an elevation of 45°, with the velocity in the last example, would be $23\frac{2}{3}$ miles, in a nonresisting medium. We shall now see what the effects will be with the resistance of the air.

PROBLEM IV.

37. *To determine the Height to which a Ball, projected Upwards as in the last Problem, will ascend, being Resisted by the Atmosphere.*

Putting x to denote any variable and increasing height ascended by the ball; v its variable and decreasing velocity there; d the diameter of the ball, its weight being w; $m =$ 00000757, and $n =$ ·00175, the coefficients of the two terms denoting the law of the air's resistance. Then $(mv^2 - nv)$ d^2, by cor. 1 to prob. 2, will be the resistance of the air against the ball in avoirdupois pounds; to which if the weight of the ball be added, then $(mv^2 - nv)d^2 + w$ will be the whole resistance to the ball's motion; this divided by w, the weight of the ball in motion, gives $\frac{(mv^2 - nv)d^2 + w}{w} =$ $\frac{mv^2 - nv}{w} d^2 + 1 = f$ the retarding force. Hence the general formula $v\dot{v} = 2gf\dot{x}$ (theor. 10 pa. 342 vol. 2 of the Course) becomes $-\dot{v}v = 2g\dot{x} \times \frac{(mv^2 - nv)d^2 + w}{w}$, making $\dot{v}$ negative because v is decreasing, where $g = 16$ feet; and hence

$$\dot{x} = -\frac{w}{2g} \times \frac{v\dot{v}}{(mv^2 - nv)d^2 + w} = \frac{-w}{2gmd^2} \times \frac{v\dot{v}}{v^2 - \frac{n}{m}v + \frac{w}{md^2}}.$$

Now, for the easier finding the fluent of this, assume $v - \frac{n}{2m} = z$; then $v = z + \frac{n}{2m}$, and $v^2 = z^2 + \frac{n}{m}z + \frac{n^2}{4m^2}$, and $v\dot{v} = z\dot{z} + \frac{n}{2m}\dot{z}$, and $v^2 - \frac{n}{m}v + \frac{n^2}{4m^2} = z^2$, and $v^2 -$

$\frac{n}{m} v = z^2 - \frac{n^2}{4m^2}$; these being substituted in the above value of $\dot{x}$, it becomes $\dot{x} =$

$$\frac{-w}{2gmd^2} \times \frac{\dot{z}z + \frac{n}{2m}\dot{z}}{z^2 - \frac{n^2}{4m^2} + \frac{w}{md^2}} = \frac{-w}{2gmd^2} \times \frac{\dot{z}z + p\dot{z}}{z^2 + \frac{w}{md^2} - p^2} = \frac{-w}{2gmd^2} \times \frac{\dot{z}z + p\dot{z}}{z^2 + q^2},$$

putting $p = \frac{n}{2m}$, and $q^2 = \frac{w}{md^2} - p^2$, or $p^2 + q^2 = \frac{w}{md^2}$.

Then the general fluents, taken by the 8th and 11th forms vol. 2 pa. 307 of the Course, give $x = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (z^2 + q^2) + \frac{p}{q^2} \times$ arc to rad. q and tan. $z] = \frac{-w}{2gmd^2} \times [\frac{1}{2} \log. (v^2 - \frac{n}{m} v + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc to rad. q. and tang. $v - p]$. But, at the beginning of the motion, when the first velocity is v for instance, and the space x is $= 0$, this fluent becomes $0 = \frac{-w}{2gmd^2} \times [\frac{1}{2}\log. (\mathrm{v}^2 - \frac{n}{m}\mathrm{v} + \frac{w}{md^2}) + \frac{p}{q^2} \times$ arc radius q tan. $\mathrm{v} - p]$. Hence by subtraction, and taking $v = 0$ for the end of the motion, the correct fluent becomes $x = \frac{w}{2gmd^2} \times [\frac{1}{2}\log. (\mathrm{v}^2 - \frac{n}{m}\mathrm{v} + \frac{w}{md^2}) - \frac{1}{2}\log. \frac{w}{md^2} + \frac{p}{q^2} \times$ (arc tan. $\mathrm{v} - p -$ arc tan. $- p$ to rad q)].

But as part of this fluent, denoted by $\frac{p}{q^2} \times$ the dif. of the two arcs to tans. $\mathrm{v} - p$ and $-p$, is always very small in comparison with the other preceding terms, it may be omitted, without material error in any practical instance; and then the fluent is $x = \frac{w}{4gmd^2} \times$ hyp. log. $\frac{\mathrm{v}^2 - \frac{n}{m}\mathrm{v} + \frac{w}{md^2}}{\frac{w}{md^2}}$, for the utmost height to which the ball will ascend, when its motion ceases, and is stopped, partly by its own gravity, but chiefly by the resistance of the air.

38. But now, for the numerical value of the general coefficient $\frac{w}{4gmd^2}$, and the term $\frac{w}{md^2}$; because the mass of the ball to the diameter d, is $\cdot 5236d^3$, if its specific gravity be s, its weight will be $\cdot 5236sd^3 = w$; therefore $\frac{w}{d^2} = \cdot 5236sd$, and $\frac{w}{md^2} = 69259sd$, this divided by $4g$ or 64, it gives $\frac{w}{4gmd^2} = 1082sd$ for the value of the general coefficient, to any diame-

ter d and specific gravity s. And if we further suppose the ball to be cast iron, the specific gravity, or weight of one cubic inch of which, is ·26855lb, it becomes $290{\cdot}6d$, for that coefficient; also $69259sd = 18600d = \frac{w}{md^2}$, and $\frac{n}{m} = 231{\cdot}5$. Hence the foregoing fluent becomes $290{\cdot}6 \times$ hyp. log. $\frac{v^2 - 231{\cdot}5v + 18600}{18600d}$, or $669d \times$ com. log. $\frac{v^2 - 231{\cdot}5v + 18600d}{18600d}$, changing the hyperbolic for the common logs. And this is a general expression for the altitude in feet, ascended by any iron ball, whose diameter is d inches, discharged with any velocity v feet. So that, substituting any values of d and v, the particular heights will be given, to which the balls will ascend.

39. *Exam.* 1. Suppose the ball be that belonging to the last table of resistances, its weight being 18 oz, or $1\frac{1}{8}$lb, and its diameter 2 inches, when discharged with the velocity 2000 feet, being nearly the greatest velocity for any iron ball. The calculation being made with these values of d and v, the height ascended is found to be 2653 feet, or only about half a mile; though found to be almost 12 miles without the air's resistance. And thus the height may be found for any other diameter and velocity.

40. *Exam.* 2. Again, for the 24lb ball, with the same velocity 2000, its diameter being nearly $5{\cdot}6 = d$. Here $669d = 3746$, and $\frac{v^2 - 231{\cdot}5v + 18600d}{18600d} = \frac{36416}{10416}$, the log. of which is 1·54354; therefore $1{\cdot}54354 \times 3746 = 5782 = x$ the height, being a little more than a mile.

We may now examine what will be the height ascended, considering the resistance always as the square of the velocity.

PROBLEM V.

41. *To determine the Height ascended by a Ball, projected as in the two foregoing Problems; supposing the Resistance of the Air to be as the Square of the Velocity.*

Here it will be proper to commence with selecting some experimented resistance corresponding to a medium kind of

velocity, between the greatest velocity and nothing, from which to compute the other general resistances, by considering them as the squares of the velocities. It is proper to assume a near medium velocity and its resistance, because, if we assume or commence with the greatest, or the velocity of projection, and compute from it downwards, the resistances will be everywhere too great, and the altitude ascended much less than just; and, on the other hand, if we assume or commence with a small resistance, and compute from it all the others upwards, they will be much too little, and the computed altitude far too great. But, commencing with a medium degree, as for instance that which has a resistance about the half of the first or greatest resistance, or rather a little more, and computing from that, then all those computed resistances above that, will be rather too little, but all those below it too great; by which it will happen, that the defect of the one side will be compensated by the excess on the other, and the final conclusion be near the truth.

42. Thus then, if we wish to determine, in this way, the altitude ascended by the ball employed in the former table of resistances, when projected with 2000 feet velocity; we perceive by the table, that to the velocity 2000 corresponds the resistance 102lb; the half of this is 51, to which resistance corresponds the velocity 1400 in the table, and the next greater velocity 1500, with its resistance 59, will be properest to be employed here. Hence then, for any other velocity v, in general, it will be, according to the law of the squares of the velocities, as $1500^2 : v^2 :: 59 : \frac{59v^2}{1500^2} = \cdot 000026\frac{2}{9}v^2 = av^2$, putting $a = \cdot 000026\frac{2}{9}$, which will denote the air's resistance for any velocity v, very nearly, counting from 2000 down to 0.

43. Now let x denote the altitude ascended when the velocity is v, and w the weight of the ball: then, as above, av^2 is the resistance from the air, hence $av^2 + w$ is the whole resisting force, and $\frac{av^2+w}{w} = f$ the retarding force;

$$\text{therefore } -\dot{vv} = 2gf\dot{x} = \frac{av^2+w}{w} \times 2g\dot{x};$$

$$\text{and hence } \dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{av^2+w} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2+\frac{w}{a}};$$

the fluent of which, by form 8, is $\frac{-w}{4ga} \times$ h. log. $(v^2 + \frac{w}{a})$; which when $x = 0$, and $v = \mathrm{v}$ the first or projectile velocity, becomes $0 = \frac{-w}{4ga} \times$ h. l. $(\mathrm{v}^2 + \frac{w}{a})$; therefore, by subtracting, the correct fluent is $x = \frac{w}{4ga} \times$ h. l. $\frac{av^2+w}{a\mathrm{v}^2+w}$, the height x when the velocity is reduced to v; and when $v = 0$, or the velocity is quite exhausted, this becomes $\frac{w}{4ga} \times$ h. l. $\frac{a\mathrm{v}^2+w}{w}$ for the whole height to which the ball will ascend.

44. *Ex.* 1. The values of the letters being $w = 1\frac{1}{3}$lb, $4g = 64$, $a = \cdot 000026\frac{2}{9}$, the last expression becomes $670 \times$ hyp. log. $\frac{\mathrm{v}^2+42880}{42880}$, or $1484 \times$ com. log. $\frac{\mathrm{v}^2+42880}{42880}$. And here the first vel. v being 2000, the same expression $1484 \times$ log. $\frac{\mathrm{v}^2+42880}{42880}$ becomes $1484 \times$ log. of $94\cdot28 = 2930$ for the height ascended, on this hypothesis; which was 2653 by the former problem.

45. *Ex.* 2. Supposing the same ball to be projected with the velocity of only 1500 feet. Then taking 1100 velocity, whose tabular resistance is 28·6, being nearest to the half of that for 1500. Hence, as $1100^2 : v^2 :: 28\cdot6 : \cdot00002364v^2 = av^2$. This value of a substituted in the theorem $\frac{w}{4ga} \times$ h. l. $\frac{av^2+w}{w}$, also 1500 for v, and $1\frac{1}{3}$ for w, it brings out $x = 2882$ for the height in this case.

46. *Ex.* 3. To find the height ascended by the same ball, projected with 820 feet velocity. Here taking 600, whose resistance 6·90 is a near medium; then as $600^2 : 6\cdot90 :: 1 : \cdot000019\frac{1}{6} = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{a\mathrm{v}^2+w}{w} = 1802$ the height in this case.

47. *Ex.* 4. With the same ball, and 1640 velocity. Assume 1200, whose resistance 35·275 is nearly a medium. Then as $1200^2 : 35\cdot275^2 :: 1 : \cdot0000245 = a$. Hence $\frac{w}{64a} \times$ h. l. $\frac{a\mathrm{v}^2+w}{w} = 2500$, the height in this case.

48. *Ex.* 5. For any other ball, whose diameter is d, and its weight w, the resistance of the air being $\frac{ad^2v^2}{4} = \frac{d^2v^2}{152542} = bd^2v^2$, putting $b = \frac{1}{152542}$, the retarding force will be $\frac{bd^2v^2 + w}{w}$, thence $-\dot{v}v = 2g\dot{x} \times \frac{bd^2v^2+w}{w}$, and $\dot{x} = \frac{-w}{2g} \times \frac{v\dot{v}}{bd^2v^2 + w}$, and the cor. flu. $x = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2v^2+w}{bd^2v^2+w} = \frac{w}{4gbd^2} \times$ h. l. $\frac{bd^2v^2+w}{w}$ for the whole height when $v = 0$. Now if the ball be a 24 pounder, whose diameter is 5·6 nearly, and its square 31·36; then $bd^2 = \cdot 00020714$, and $\frac{w}{4gbd^2} = \frac{24}{64bd^2} = \frac{3}{8bd^2} = 1810$; also $v = 2000$, $bd^2v^2 = 829$, and $\frac{bd^2v^2+w}{w} = \frac{829+24}{24} = \frac{853}{24}$; therefore $x = 1810 \times$ h. l. $\frac{853}{24} = 1810 \times 3\cdot57071 = 6463$, being more than double the height of that of the small ball, or a little more than a mile, and nearly the same as in the 2d example to prob. 4.

PROBLEM VI.

49. *To ascertain the Time of the Ball's ascending to the Height determined in the last Problem, by the same Projectile Velocity as there given.*

By that prob. $\dot{x} = \frac{-w}{2ga} \times \frac{v\dot{v}}{v^2 + \frac{w}{a}}$, therefore $\dot{t} = \frac{\dot{x}}{v} = \frac{-w}{2ga} \times \frac{\dot{v}}{v^2 + \frac{w}{a}}$; the fluent of which, by form 11, is $\frac{-w}{2ga}\sqrt{\frac{a}{w}} \times$ arc to radius 1 tang. $\frac{v}{\sqrt{\frac{w}{a}}} = \frac{-1}{2g}\sqrt{\frac{w}{a}} \times$ arc tan. $\frac{v}{\sqrt{\frac{w}{a}}}$; or by correction $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ (arc tang. $\frac{v}{\sqrt{\frac{w}{a}}}$ − arc tang. $\frac{v}{\sqrt{\frac{w}{a}}}$), the time in general when the first velocity v is reduced to v. And when $v = 0$, or the velocity ceases, this becomes $t = \frac{1}{2g}\sqrt{\frac{w}{a}} \times$ arc to tang. $\frac{v}{\sqrt{\frac{w}{a}}}$ for the time of the whole ascent; which is also equal to $\frac{1}{2g} \times$ arc to tang. v and rad. $\sqrt{\frac{w}{a}}$.

50. But $\frac{1}{2g} \times v$ or $\frac{v}{32}$ denotes the time of the body's whole ascent in free space, projected up with the same velocity v. Therefore the whole time of ascent in free space, is to the whole time of ascent in the atmosphere, commencing with the same velocity, as the tangent of v, is to its arc, the radius being $\sqrt{\frac{w}{a}}$.

51. Now, as in the last prob. $v = 2000$, $w = 1\frac{1}{8}$, $a = \cdot 000026\frac{2}{9} = \frac{236}{9000000}$. Hence $\frac{w}{a} = 42900$, and $\sqrt{\frac{w}{a}} = 207$, and $\frac{v}{\sqrt{\frac{w}{a}}} = 9\cdot 66184$ the tangent, to which corresponds the arc of 84° 6′, whose length is 1·4678; then $\frac{1}{2g} \times 207 \times 1\cdot 4678 = \frac{207 \times 14678}{32} = 9''\cdot 5$, the whole time of ascent.

52. *Corol.* 1. The time of *freely* ascending or descending through the same height 2800 feet, nearly the mean between the two former results, that is, without the air's resistance, would be $\sqrt{\frac{2800}{16}} = \frac{53}{4} = 13''\frac{1}{4}$; and the time of *freely* ascending, till all the velocity is lost, commencing with the same velocity 2000, would be $\frac{v}{2g} = \frac{2000}{32} = 62''\frac{1}{2} = 1'\,2''\frac{1}{2}$, being in proportion to $9''\frac{1}{2}$ the time as above in the atmosphere, as 9·6618 the tangent of 84°6′, to 1·4678 its arc. But the time of ascending *freely* through the space 2930 only, commencing with the same velocity 2000, would be only $1''\frac{1}{2}$.

53. *Corol.* 2. Since the time of ascent is in general equal to $\frac{1}{2g}\sqrt{\frac{w}{a}}$ arc tang. $v\sqrt{\frac{a}{w}}$, to radius 1, nearly, the greatest possible time, when v is infinite, or the tangent $v \times \sqrt{\frac{a}{w}}$ is infinite, or when the arc is a quadrant, will be nearly $\frac{1}{2g}\sqrt{\frac{w}{a}} \times$ the quadrant to the radius 1, that is, $\frac{c}{4g}\sqrt{\frac{w}{a}}$, where $c = 3\cdot 1416$, the circumference of the circle to the diameter 1. Now, with the given numbers, viz, $a = \cdot 000026\frac{2}{9}$, $c = 3\cdot 1416$, $g = 64$, the greatest time $\frac{c}{4g}\sqrt{\frac{w}{a}}$ becomes $9\cdot 6 \times \sqrt{10}$, which therefore is the greatest time nearly. For the

$1\frac{1}{3}$lb ball, when $w = 1\frac{1}{3}$, or $\sqrt{w} = 1{\cdot}06$, this time $9{\cdot}6\sqrt{w}$, is $10{\cdot}176$ seconds.

54. *Corol.* 3. For any other size of ball, as that of d diameter, instead of the one of 2 inches diameter, employed above, we must take, in the theorem, $\frac{1}{4}d^2a$ instead of a, and then the whole time in this general case will be $\frac{2}{d} \times \frac{c}{4g}\sqrt{\frac{w}{a}}$ $= \frac{2}{d} \times 9{\cdot}6\sqrt{w} = \frac{19{\cdot}2}{d}\sqrt{w}$. But w, the ball's weight, is as d^3, the cube of the diameter, therefore $\frac{\sqrt{w}}{d}$ is as $\frac{\sqrt{d^3}}{d}$ or as $\sqrt{d}$; that is, the whole time, for the ascent of different balls, is as $\sqrt{d}$, the square-root of the diameter. Thus, for a 24 pound ball, having its diameter almost 5·6, the ratio is $\sqrt{2}$ to $\sqrt{5{\cdot}6}$, or 1 to $\sqrt{2{\cdot}8} = 1{\cdot}6733$; then $1{\cdot}6733 \times 10{\cdot}176$ $= 17$ seconds nearly, the whole time of the 24 pound ball's ascent, when projected with an infinite velocity.

PROBLEM VII.

55. *To determine the Time of a Ball's ascending to its greatest Height, using the same Formula of Resistance as in Prob.* 4.

Now, as in that problem, $\dot{x} = -\frac{w}{2g} \times \frac{v\dot{v}}{(mv^2 - nv)\,d^2 + w} =$ $\frac{-w}{2gmd^2} \times \frac{(z+p)\,\dot{z}}{z^2 + q^2}$; then dividing by v, or $z + p$, $\frac{\dot{x}}{v} = \dot{t} =$ $\frac{-w}{2gmd^2} \times \frac{\dot{z}}{z^2 + q^2}$; the general fluent of which is $t = \frac{-w}{2gmqd^2}$ $\times$ arc to tang $\frac{z}{q}$ or $\frac{v-p}{q}$ and radius 1; or, by correction, $t =$ $\frac{w}{2gmqd^2} \times$ (arc to tang. $\frac{v-p}{q}$ $-$ arc to tang. $\frac{v-p}{q}$). But, when the first velocity v is great, the arc to this latter tangent $\frac{v-p}{q}$ may be omitted, as equal to nothing, or as of no effect, since the value of p is $115\frac{1}{2}$, and when the small velocity v is about 100 or 200, the resistance, by the formula $\cdot 00003026v^2 -$ $\cdot 007v$, here used, comes out either nothing, or a small quantity negative. The latter arc being rejected then, there remains only $\frac{w}{2gmqd^2} \times$ arc to tangent $\frac{v-p}{q}$ to radius 1.

56. *For Example.* With the $1\frac{1}{2}$lb ball of 2 inches diameter, and 2000 feet velocity; that is $w = 1\frac{1}{2}$ or 1·125, $d = 2$; $v = 2000$; also $m = \cdot 00000757$, $n = \cdot 00175$; then $\frac{n}{m} = 231$, and $\frac{n}{2m} = 115\frac{1}{2} = p$, and $\frac{vv}{md^2} = 37153 = p^2 + q^2$, and $q = \sqrt{(37153 - p^2)} = 154\frac{1}{3}$; therefore $\frac{w}{2gmqd^2} = 7\cdot 522$. Also $\frac{v-p}{q} = 12\cdot 21$, the tangent to the arc of 85° 19′, the length of which is 1·4890. Then 7·522 + 1·489 = 11·199 seconds, the time of ascending by this rule. And when the velocity v is infinite, then the tangent $\frac{v-p}{q}$ becomes infinite also; consequently its arc is a quadrant 1·570, and therefore 7·522 × 1·57 = 11·81 seconds, is the time of ascent with an infinite velocity.

57. *Exam. 2.* For any other weight of ball, as suppose the 24lb $= w$, the diameter being 5·6, or more nearly 5·546 $= d$. Here then $d^2 = 30\cdot 758$, $\frac{w}{d^2} = \cdot 78029$, $\frac{w}{md^2} = 103078 = p^2 + q^2$, $p = 115\frac{1}{2}$ as before, $q = \sqrt{(103078 - p^2)} = 299\cdot 4$; $\frac{w}{2gmqd^2} = \frac{p^2+q^2}{2gq} = 10\cdot 76$. If the velocity $v = 2000$ as before, then the tangent $\frac{v-p}{q} = 6\cdot 293$, the arc of which contains 80° 58′, the length of which is 1·413; then 10·76 × 1·413 = 15·20 seconds, is the whole time of ascent when projected with 2000 feet velocity. Also 10·76 × 1·570 = 16·89 seconds, is the time when projected with an infinite velocity.

PROBLEM VIII.

58. *To determine the same as in Prob. V, taking into the account the Decrease of Density in the Air, as the Ball ascends in the Atmosphere.*

In the preceding problems, relating to the height and time of balls ascending in the atmosphere, the decrease of density in the upper parts of it has been neglected, the whole height ascended by the ball being supposed in air of the same density as at the earth's surface. But it is well known that the

atmosphere must and does decrease in density upwards, in a very rapid degree; so much so indeed, as to decrease in geometrical progression, at altitudes which rise only in arithmetical progression; by which it happens, that the altitudes ascended are proportional only to the logarithms of the decrease of density there. Hence it results, that the balls must be always less and less resisted in their ascent, with the same velocity, and that they must consequently rise to greater heights before they stop. It is now therefore to be considered what may be the difference resulting from this circumstance.

59. Now, the nature and measure of this decreasing density, of ascents in the atmosphere, has been explained and determined in prop. 76, pa. 244, &c, vol. 2 of the Course. It is there shown, that if D denote the air's density at the earth's surface, and d its density at any altitude a, or x; then is $x = 63551 \times$ log. of $\frac{D}{d}$ in feet, when the temperature of the air is 55°; we may therefore assume for the medium $x = 63500 \times$ log. $\frac{D}{d}$ for a mean degree.

60. But, to get an expression for the density d, in terms of x, out of logarithms, without which it could not be introduced into the measure of the ball's resistance, in a manageable form; we find in the first place, by a neat approximate expression for the natural number to the log. of a ratio, $\frac{D}{d}$, whose terms do not greatly differ, invented by Dr. Halley, and explained in the Introduction to our Logarithms, p. 110, that $\frac{n - \frac{1}{2}l}{n + \frac{1}{2}l} \times$ D nearly, is the number answering to the log. l of the ratio $\frac{D}{d}$, where n denotes the modulus ·43429448 &c. of the common logarithms. But, we before found that $x = 63500 \times$ log. of $\frac{D}{d}$, or $\frac{x}{63500}$ is the log. of $\frac{D}{d}$, which log. was denoted by l in the expression just above, for the number whose log. is l or $\frac{x}{63500}$; substituting therefore $\frac{x}{63500}$ for l, in the expression $\frac{n - \frac{1}{2}l}{n + \frac{1}{2}l} \times$ D, it gives the na-

tural number $\frac{n - \frac{x}{127000}}{n + \frac{x}{127000}} \times \text{D} = d$, or $\frac{127000n - x}{127000n + x} = d$, the density of the air at the altitude x, putting $\text{D} = 1$ the density at the surface. Now put $127000n$ or nearly $55000 = c$; then $\frac{c - x}{c + x}$ will be the density of the air at any general height x.

61. But, in the 5th prob. it appears that av^2 denotes the resistance to the velocity v, or at the height x, for the density of air the same as at the surface, which is too great in the ratio of $c + x$ to $c - x$; therefore $av^2 \times \frac{c - x}{c + x}$ will be the resistance at the height x, to the velocity v, where $a = \cdot 000026\frac{2}{3}$. To this adding w, the weight of the ball, gives $av^2 \times \frac{c - x}{c + x} + w$ for the whole resistance, both from the air and the ball's mass; consequently $\frac{av^2}{w} \times \frac{c - x}{c + x} + \frac{w}{w}$ will denote the accelerating force of the ball. Or, if we include the small part $\frac{w}{w}$ or 1, within the factor $\frac{c - x}{c + x}$, which will make no sensible difference in the result, but be a great deal simpler in the process, then is $\frac{av^2 + w}{w} \times \frac{c - x}{c + x} = f$ the accelerating force. Consequently $-\dot{v}v = 2gf\dot{x} = 2g\dot{x} \times \frac{c - x}{c + x} \times \frac{av^2 + w}{w}$, and hence $\frac{c - x}{c + x}\dot{x} = \frac{w}{2g} \times \frac{-v\dot{v}}{av^2 + w}$, or by division, $-\dot{x} + \frac{2c}{c + x}\dot{x} = \frac{w}{32a} \times \frac{-v\dot{v}}{v^2 + \frac{w}{a}}$.

62. Now the fluent of the first side of this equation is evidently $-x + 2c \times \text{h. l.}(c + x)$; and the fluent of the latter side, the same as in prob. 5, is $\frac{-w}{64a} \times \text{h. l.}(v^2 + \frac{w}{a})$; therefore the general fluential equation is $-x + 2c \times \text{h. l.}(c + x) = \frac{-w}{64a} \times \text{h. l.}(v^2 + \frac{w}{a})$. But, when $x = 0$, and $v = \text{v}$ the initial velocity, this becomes $0 + 2c \times \text{h. l.}\, c = \frac{-w}{64a} \times \text{h. l.}(\text{v}^2 + \frac{w}{a})$; therefore by subtraction, the correct fluents are $-x + 2c \times \text{h. l.}\frac{c + x}{c} = \frac{w}{64a} \times \text{h. l.}\frac{av^2 + w}{a\text{v}^2 + w}$, when the

first velocity v is diminished to any less one v; and when it is quite extinct, the state of the fluents becomes $-x+2c\times$ h. l. $\frac{c+x}{c}=\frac{w}{64a}\times$ h. l. $\frac{av^2+w}{w}$, for the greatest height x ascended.

63. Here, in the quantity h. l. $\frac{c+x}{c}$, the term x is always small in respect of the other term c; therefore, by the nature of logarithms, the h. l. of $\frac{c+x}{c}$ is nearly $=\frac{x}{c+\frac{1}{2}x}$ or $\frac{2x}{2c+x}$; therefore the above fluents become $-x+\frac{4cx}{2c+x}=\frac{2cx-x^2}{2c+x}=\frac{2c-x}{2c+x}x=\frac{w}{64a}\times$ h. l. $\frac{av^2+w}{w}$. Now the latter side of this equation is the same value for x as was found in the 5th problem, which therefore put $=b$; then the value of x will be easily found from the formula $\frac{2c-x}{2c+x}x=b$, by a quadratic equation. Or, still easier, and sufficiently near the truth, by substituting b for x in the numerator and the denominator of $\frac{2c-x}{2c+x}$, then $\frac{2c-b}{2c+b}x=b$, and hence $x=\frac{2c+b}{2c-b}b$, or by proportion, as $2c-b:2c+b::b:x$; that is, only increase the value of x, found by prob. 5, in the ratio of $2c-b$ to $2c+b$.

64. Now, in the first example to that prob. the value of x or b was there found $=2930$; and $2c$ being $=110000$, therefore $2c-b=107070$, and $2c+b=112930$; then, as $107070:112930::2930:3093$ = the value of the height x in this case, being only 163 feet, or $\frac{1}{18}$th part more than before.

But, for the 5th example to the 5th prob. where x was $=6463$; it will be, as $2c-b:2c+b$, or as $103537:116463::6463:7285$ the height ascended in this example, being about the 8th part more than before. And so on, for any other examples; the value of $2c$ being the constant number 110000.

Note. In a similar example to this, at the top of pa. 289 vol. 3 of the Course, there are some errors in the numbers, by having used, in the expressions $2c-b$ and $2c+b$, in the 2d line, the number 2955 instead of 6420, for the value of b.

PROBLEM IX.

65. *To determine the Time of a Ball's ascending, considering the decreasing Density of the Air as in the last Prob.*

The fluxion of the time is $\dot{t} = \frac{\dot{x}}{v}$. But the general equations of the fluxions of the space x and velocity v, in the last prob. was $\frac{c-x}{c+x}\dot{x} = \frac{w}{32} \times \frac{-v\dot{v}}{av^2+w}$; therefore $\dot{x} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-v\dot{v}}{av^2+w}$; hence $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{w}{32} \times \frac{c+x}{c-x} \times \frac{-\dot{v}}{av^2+w}$. But x, which is always small in respect of c, is nearly $= b$ as determined in the last problem; therefore $\frac{c+b}{c-b}$ may be substituted for $\frac{c+x}{c-x}$ without sensible error; and then $\dot{t}$ becomes $= \frac{w}{32} \times \frac{c+b}{c-b} \times \frac{-\dot{v}}{av^2+w}$. Now, this fluxion being to that in prob. 6, in the constant ratio of $c - b$ to $c + b$, their fluents will be also in the same constant ratio. But, by the last prob. $c = 55000$, and $b = 2930$ for the first example in prob. 5; therefore $c - b = 52070$, and $c + b = 57930$, also the time in problem 6 was $9''\cdot5$; therefore, as $52070 : 57930 :: 9''\cdot5 : 10''\cdot57$ for the time in this case, being $1''\cdot07$ more than the former, or nearly the 9th part more; which is nearly the double, or as the square of the difference, in the last prob. in the height ascended.

PROBLEM X.

66. *To determine the circumstances of Space, Time, and Velocity, of a Ball descending through the Atmosphere by its own Weight.*

It is here meant that the balls are at least as heavy as cast iron, and therefore their loss of weight in the air insensible; and that their motion commences by their own gravity from a state of rest. The first object of enquiry may be, the utmost degree of velocity any such ball acquires by thus descending. Now it is manifest that the ball's motion is commenced, and uniformly increased, by its own weight, which is its constant

urging force, being always the same, and producing an equal increase of velocity in equal times, excepting for the diminution of motion by the air's resistance. It is also evident that this resistance, beginning from nothing, continually increases, in some ratio, with the increasing velocity of the ball. Now, as the urging force is constantly the same, and the resisting force always increasing, it must happen that the latter will at length become equal to the former: when this obtains, there can afterwards be no further acceleration of the motion, the impelling force and the resistance being equal, and the ball must ever after descend with a uniform motion. It follows therefore that, to answer the first enquiry, we have only to determine when or what velocity of the ball will cause a resistance just equal to its own weight.

67. Now, by inspecting the table of resistances preceding prob. 1, or in prob. 2, the weight of the ball being $1\frac{1}{8}$lb, we perceive that the resistance increases in the last column, till 0·709 opposite to 200 velocity, and 1·612 answering to 300 velocity, between which two the proposed resistance 1·125, and the correspondent velocity, fall. But, in two velocities not greatly different, the resistances are very nearly proportional to the squares of the velocities. Therefore, having given the velocity 200 answering to the resistance 0·709, to find the velocity answering to the resistance 1·125, we must say, as $0{\cdot}709 : 1{\cdot}125 :: 200^2 : v^2 = 63470$, therefore $v = \sqrt{63470} = 252$, is the greatest velocity this ball can acquire; after which it will descend with that velocity uniformly, or at least with a velocity nearly approaching to 252.

The same greatest or uniform velocity will also be directly found from the rule $\cdot 0000176 v^2 = r$, near the end of problem 2, where r is the resistance to the velocity v, by making $1{\cdot}125 = r$; for then $v^2 = \frac{1{\cdot}125}{\cdot 0000176} = 63920$, the root of which is 253, the same value as before nearly.

68. But now, for any other weight of ball; since the weights of the balls increase as the cubes of their diameters, and their resistances, being as the surfaces, increase only as

the squares of the same, which is one power less; and the resistances being also, in this case, as the squares of the velocities, we must therefore increase the squares of the velocity in the ratio of the diameters of the balls; that is, as $2 : d :: 252^2 : 31752d = v^2$, and hence $v = 178\sqrt{d}$, is a general value of the greatest velocity for any ball whose diameter is d inches.

69. If we take here the 3lb ball, belonging to the second table of resistances, nearly, the diameter d being $= 2{\cdot}773$; then $\sqrt{2{\cdot}773} = 1{\cdot}666$, and $178 \times 1{\cdot}666 = 297$, is the greatest or uniform velocity, with which the 3lb ball will descend. And if we take the 6lb ball, whose diameter is 3·494 inches, as in the third table of resistances nearly: then $\sqrt{3{\cdot}494} = 1{\cdot}870$, and $178 \times 1{\cdot}870 = 333$, being the greatest velocity that can be acquired by the 6lb ball, and with which it will afterwards uniformly descend. For a 9lb ball, whose diameter is 4·00, the velocity will be $178 \times 2 = 356$. And so on for any other size of iron ball, as in the following table.

Wt. lbs.	Diam. inches	Termin. Velocity v, feet	Height due to v, feet	Times of freely falling
1	1·923	247	948	7·72
2	2·423	277	1193	8·66
3	2·773	297	1371	9·28
4	3·053	311	1503	9·72
6	3·494	333	1724	10·41
9	4·000	356	1970	11·12
12	4·403	374	2174	11·69
18	5·040	400	2488	12·50
24	5·546	419	2729	13·09
32	6·106	440	3010	13·75
36	6·350	449	3134	14·03
42	6·684	461	3304	14·50

Where the first column contains the weight of the balls in lbs; the 2d their diameters in inches; the 3d their velocities to which they nearly approach, as a limit, and therefore called their terminal or last velocities, with which they afterward descend uniformly; and the 4th column the heights due to

those velocities, or the heights from which the balls must descend in vacuo to acquire them. Also the 5th or last column shows the times of descending through these heights freely, or without resistance.

But it is manifest that the balls can never attain exactly to these velocities in any finite time or descent, being only the limits to which they continually approach, without ever really reaching, though they arrive very nearly at them in a short space of time; as will appear hereafter.

70. The greatest velocity being $178\sqrt{d}$ feet, when d is expressed in inches, it will be $178\sqrt{12d}$ or $616{\cdot}6\sqrt{d}$, where d is expressed in feet. Now, the spaces through which a body must freely descend by gravity, to acquire the same velocity, will be thus found: as $32 : \sqrt{16} :: 616{\cdot}6\sqrt{d} : \sqrt{h}$, or $8 : 1 :: 616{\cdot}6\sqrt{d} : \sqrt{h}$, or $77{\cdot}1\sqrt{d} = \sqrt{h}$, hence $h = 5944d$. But the density of the ball is to the density of the air, as 7400 to $1\frac{2}{9}$ nearly, or as 66600 to 11, or as 6054 to 1. And 6054 to 1 is nearly the same ratio as $5944d$ or h to d. Therefore the greatest velocity with which a ball can descend in the air, is the same which it may acquire by falling without resistance, and in its fall describing a space that is in proportion to the ball's diameter, as the density of the ball to the density of the air, very nearly.

The exact spaces, h or $5944d$, answering to the velocities v in the 3rd column, are set in the 4th column of the table above. Also the times of freely falling in vacuo through those spaces, or of acquiring the velocities in the 3rd column, are placed in the 5th or last column, being found by dividing the greatest velocity v by 32 feet, the velocity acquired in the first second of time, viz, $\frac{v}{32} = t$ seconds, the time.

71. To obtain general expressions for the space descended, and the time of the descent, in terms of the velocity v: put x = any space descended, t = its time, and v the velocity acquired, the weight of the ball $w = 1\frac{1}{8}$lb. Now, by the theorem near the end of prob. 2, which is the proper rule for this case, the velocity being small, $\cdot 0000176v^2 = cv^2$ is the resistance due to the velocity v; therefore $w - cv^2$ is the impelling

force, and $\frac{w - cv^2}{w} = f$ the accelerating force; conseq. $\dot{v}v$ or $2gf\dot{x} = 2g\dot{x} \times \frac{w - cv^2}{w}$, and $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, the correct fluent of which, by the 8th form, is $x = \frac{w}{4gc} \times$ h. l. $\frac{w}{w - cv^2}$, the general value of the space x descended.

72. Here it appears that the denominator $w - cv^2$ decreases as v increases; consequently the whole value of x, the descent, increases with v, till it becomes infinite, when the resistance cv^2 is $= x$ the weight of the ball, and when the motion becomes uniform, as before remarked.

73. We may however easily assign the value of x a little before the velocity becomes uniform, or before cv^2 becomes $= w$. Thus, when $cv^2 = w$, then $v = 252$, as found in the beginning of this problem. Assume therefore v a little less than that greatest velocity, as for instance 246: then this value of v substituted in the general formula for x above deduced, gives $x = 2927$ feet, a little before the motion becomes uniform, or when the velocity has arrived at 246, its maximum being 252.

74. In like manner is the space to be computed that will be due to any other velocity, less than the greatest or terminal velocity. On the contrary, to find the velocity due to any proposed space x, from the formula $x = \frac{w}{4gc} \times$ h. l. $\frac{w}{w - cv^2}$. Here x is given, to find v. First then $\frac{4gcx}{w} =$ h. l. $\frac{w}{w - cv^2}$; take therefore the number to the hyp. log. of $\frac{4gcx}{w}$, which number call N; then $N = \frac{w}{w - cv^2}$; conseq. $Nw - NCv^2 = w$, and $Nw - w = NCv^2$, and $v = \sqrt{\frac{N - 1}{NC}w}$, a general theorem for the value of v due to any space x. Suppose, for instance, x is 1000. Now $4g$ being $= 64$, $w = 1\frac{1}{8}$, and $c = \cdot 0000176$; therefore $\frac{4gcx}{w} = 1\cdot 0012$, and the natural number belonging to this, considered as an hyp. log. is $2\cdot 72162 = N$; hence then $v = \sqrt{\frac{N - 1}{NC}w} = 201$, is the velocity due to the space 1000, or when the ball has descended 1000 feet.

75. Again, for the time t of descent: here $\dot{t} = \frac{\dot{x}}{v}$; but $\dot{x} = \frac{w}{2g} \times \frac{v\dot{v}}{w - cv^2}$, as found above, therefore $\dot{t} = \frac{w}{2g} \times \frac{\dot{v}}{w - cv^2}$, the fluent of which is $\frac{1}{4g}\sqrt{\frac{w}{c}} \times \text{h. l.} \frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}$, the general value of the time t for any value of the velocity v; which value of t evidently increases as the denominator $\sqrt{\frac{w}{c}} - v$ decreases, or as the velocity v increases; and consequently the time is infinite when that denominator vanishes, which is when $v = \sqrt{\frac{w}{c}}$, or $cv^2 = w$, the resistance equal to the ball's weight, being the same case as when the space x becomes infinite, as above remarked.

76. But, like as was done for the distance x as above, we may here also find the value of t corresponding to any value of v, less than its maximum 252, and consequently to any value of x, as when v is 246 for instance, or $x = 2927$, as determined above. Now, by substituting 246 for v, in the general formula

$t = \frac{1}{4g}\sqrt{\frac{w}{c}} \times \text{h. l.} \frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v}$, it brings out $t = 16''\cdot957$; so that it would be nearly 17 seconds when the velocity arrives at 246, or a little less than the maximum or uniform degree, viz, 252, or when the space descended is 2927 feet.

77. Also, to determine the time corresponding to the same, when the descent is 1000 feet, or the velocity 201: find the value of $\frac{1}{4g}\sqrt{\frac{w}{c}} = \frac{1}{64}\sqrt{\frac{1\cdot125}{\cdot0000176}} = \frac{252}{64} = \frac{63}{16}$. Then

$\frac{\sqrt{\frac{w}{c}} + v}{\sqrt{\frac{w}{c}} - v} = \frac{252 + 201}{252 - 201} = \frac{453}{51}$; the hyp. log. of which is $2\cdot18407$.

Hence $2\cdot18407 \times \frac{63}{16} = 8''\cdot6$, the time of descending 1000 feet, or when the velocity is 201.

See other speculations on this problem, in the 2d volume of the Course, prob. 22, as determined from theory, viz, without using the experimented resistance of the air.

PROBLEM XI.

78. *To determine the Circumstances of the Motion of a Ball projected horizontally in the Air; abstracted from its Vertical Descent by its Gravitation, or supposing the Ball void of Gravity.*

Putting d for the diameter, and w the weight of the ball, v the velocity of projection, and v the velocity of the ball after having moved through the space x. Then, by corol. 1 to prob. 2, if the velocity is considerable, such as usual in practice, the resistance of the ball, moving with the velocity v, is $(mv^2 - nv)d^2$, and therefore $\frac{mv^2 - nv}{w}d^2$, is the retardive force f; hence the common formula $v\dot{v} = 2gf\dot{x}$, is $-v\dot{v} = 32\dot{x} \times \frac{mv^2 - nv}{w}d^2$, and theref. $\dot{x} = \frac{w}{32d^2} \times \frac{-v\dot{v}}{mv^2 - nv} = \frac{w}{32d^2} \times \frac{-\dot{v}}{mv - n} = \frac{w}{32d^2m} \times \frac{-\dot{v}}{v - \frac{n}{m}}$, the fluent of which is obviously $\frac{w}{32md} \times -$ hyp. log. of $v - \frac{n}{m}$; and by the correction by the first velocity v, it becomes $x = \frac{w}{32md^2} \times$ h. log. $\frac{\mathrm{v} - \frac{n}{m}}{v - \frac{n}{m}}$ $= \frac{w}{32md^2} \times$ h. log. $\frac{\mathrm{v} - q}{v - q}$, the general formula for the distance passed over in terms of the velocity, where $q = \frac{n}{m} = \frac{\cdot 00175}{\cdot 000007565} = 231$.

79. But now to reduce the general coefficient $\frac{w}{32md^2}$ to a simple number in terms of d the diameter only; the weight $w = \cdot 5236d^3 \times 4\cdot 3 = 2\frac{1}{4}d^3 = \frac{9}{4}d^3$, where $4\cdot 3$ is the weight in ounces of 1 cubic inch of cast iron, or $w = \frac{9}{64}d^3$ in pounds weight. Therefore $\frac{w}{d^2} = \frac{9}{64}d$, and $\frac{w}{32md^2} = 581\frac{1}{4}d$; then the theorem becomes $x = 581\frac{1}{4}d \times$ h. log. $\frac{\mathrm{v} - q}{v - q}$; or, multiplying the coefficient by $2\cdot 30258$ the hyp. log. of 10, to reduce the hyp. log. to the common log. then it is $x = 1338d \times$ com. log. of $\frac{\mathrm{v} - q}{v - q}$, the diameter d being in inches.

80. Now, for an application, let it be required first, to determine in what space a 24lb ball will have its velocity reduced from 1780 feet to 1500, that is, losing 280 feet of its first velocity. Here $d = 5{\cdot}546$, $w = 24$, $\mathrm{v} = 1780$, and $v = 1500$; also $\frac{n}{m} = 231$. Hence $x = 7420 \times \log. \frac{\mathrm{v} - 231}{v - 231} = 7420{\cdot}4 \times \log. \frac{1549}{1269} = 642$ feet, the space passed over when the ball has lost 280 feet of its motion.

81. Again, to find with what velocity the same ball will move, after having described 1000 feet in its flight. The above theorem is x or $1000 = 7420 \times \log. \frac{\mathrm{v} - 231}{v - 231}$, or $\frac{1000}{7420}$ = the common log. of $\frac{1549}{v-231}$; but the number to the com. log. $\frac{1000}{7420}$ is $1{\cdot}3635 = \mathrm{N}$ suppose; then $\mathrm{N} = \frac{1549}{v-231}$, and $\mathrm{N}v - 231\mathrm{N} = 1549$, or $\mathrm{N}v = 1549 + 231\mathrm{N}$, and $v = \frac{1549}{\mathrm{N}} + 231 = 1136 + 231 = 1367$, the velocity when the ball has moved 1000 feet.

82. Next, to find a theorem for the time of describing any space, or destroying any velocity.

Here $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32md^2} \times \frac{-v^{-1}\dot{v}}{v - \frac{n}{m}} = \frac{w}{32md^2} \times \frac{-v^{-1}\dot{v}}{v-q}$, the fluent of which, by the 9th form, is $t = \frac{w}{32md^2} \times \frac{1}{q} \times \mathrm{h.l.} \frac{v}{v-q} = \frac{w}{32md^2q} \times \mathrm{h.l.} \frac{v}{v-q}$, and by correction

$t = \frac{w}{32md^2q} \times (\mathrm{h.l.} \frac{v}{v-q} - \mathrm{h.l.} \frac{\mathrm{v}}{\mathrm{v}-q}) = \frac{w}{32md^2q} \times$ hyp. log. $\frac{\mathrm{v}-231}{v-231} \cdot \frac{v}{\mathrm{v}} = \frac{1338d}{231} \times$ com. log. $\frac{\mathrm{v}-q}{v-q} \cdot \frac{v}{\mathrm{v}}$, putting v for the first velocity, and 231 for $\frac{n}{m}$ or q its value, as before.

83. Now, to take for an example the same 24lb ball, and its projected velocity 1780, as before; let it be required to find in what time this velocity will be reduced to 786. Here then $\mathrm{v} = 1780$, $v = 786$, $w = 24$, $d = 5{\cdot}546$, $d^2 = 30{\cdot}76$; hence $\frac{1338d}{231} = \frac{7420}{231} = 32\frac{1}{8}$; and $\frac{\mathrm{v} - 231}{v - 231} \cdot \frac{v}{\mathrm{v}} = \frac{1549}{555} \times \frac{786}{1780} = 1{\cdot}232$, the log. of which is ·0906; then $32\frac{1}{8} \times {\cdot}0906 = 2''{\cdot}9$, the time required, being almost 3 seconds.

84. For another example, let it be required to find when the velocity will be reduced to 1000, or 780 destroyed. Here $v = 1000$, and all the other quantities as before. Then $\frac{\mathrm{v} - 231}{v - 231} \times \frac{v}{\mathrm{v}} = \frac{1549}{769} \times \frac{1000}{1780} = \frac{1549}{1369}$, the log. of which is ·053648; theref. $32\frac{1}{8} \times$ ·053648 = 1·723 seconds, is the time sought.

85. On the other hand, if it be required to find what will be the velocity after the ball has been in motion during any given time, as suppose 2 seconds, we must reverse the calculation thus: $t = 2''$ being $= 32\frac{1}{8} \times$ log. $\frac{\mathrm{v} - 231}{v - 231} \cdot \frac{v}{\mathrm{v}}$; theref. $\frac{2}{32\frac{1}{8}} =$ ·062218 is the log. of $\frac{\mathrm{v} - 231}{v - 231} \cdot \frac{v}{\mathrm{v}}$, the number answering to which is 1·1536 = N suppose, that is, $\mathrm{N} = \frac{\mathrm{v} - 231}{v - 231} \cdot \frac{v}{\mathrm{v}}$. Hence $\mathrm{N}\mathrm{v}v - 231\mathrm{N}\mathrm{v} = \mathrm{v}v - 231v$, and $v = \frac{231\mathrm{N}\mathrm{v}}{231 + \mathrm{N}\mathrm{v} - \mathrm{v}} = \frac{474337}{504} = 941$, the velocity at the end of 2 seconds.

86. The foregoing calculations serve only for the higher velocities, such as exceed 300 or 400 feet per second of time. But, for those that are below 400, the rule is simpler, as the resistance is then, by cor. 2 prob. 2, $\cdot 0000044 d^2 v^2 = cd^2v^2$, where d denotes the diameter of any ball. Hence then, employing the same notation as before, $\frac{cd^2v^2}{w} = f$, and $-v\dot{v} = 32f\dot{x} = 32\dot{x} \times \frac{cd^2v^2}{w}$; therefore $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, the correct fluent of which is $x = \frac{w}{32cd^2} \times$ h. l. $\frac{\mathrm{v}}{v}$.

87. Now, for an example, suppose the first velocity to be $300 = \mathrm{v}$, and the last $v = 100$, for a 24lb ball. Then $w = 24$, $d = 5·546$, $d^2 = 30·76$, $c = \cdot 0000044$; therefore $\frac{w}{32cd^2} = \frac{3}{123·04c} = 5542$; and $\frac{\mathrm{v}}{v} = \frac{300}{100} = 3$, the hyp. log. of which is 1·0986; therefore 1·0986 × 5542 = 6108 = x, is the distance. If the first velocity be only 200 = v; then $\frac{\mathrm{v}}{v} = 2$, the hyp. log. of which is ·69315, therefore ·69315 × 5542 = 3841 = x, the distance.

88. And conversely, to find what velocity will remain after passing over any space, as 4000 feet, the first velocity being $\mathrm{v} = 200$. Here the hyp. log. of $\frac{\mathrm{v}}{v}$ is $\frac{x}{5542} = \frac{4000}{5542} =$ ·72178, the natural number of which is 2·06 nearly, that is, $2{\cdot}06 = \frac{\mathrm{v}}{v}$; therefore $v = \frac{\mathrm{v}}{2{\cdot}06} = \frac{200}{2{\cdot}06} = 97{\cdot}1$, the velocity.

89. Again, for the time t: since $\dot{x} = \frac{w}{32cd^2} \times \frac{-\dot{v}}{v}$, therefore $\dot{t} = \frac{\dot{x}}{v} = \frac{w}{32dc^2} \times \frac{-\dot{v}}{v^2}$, the correct fluent of which is $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{\mathrm{v}}) = \frac{w}{32cd^2} \times \frac{\mathrm{v}-v}{\mathrm{v}v}$.—So, for example, if $\mathrm{v} = 300$, and $v = 100$; then $\frac{\mathrm{v}-v}{\mathrm{v}v} = \frac{200}{30000} = \frac{2}{300}$; and $\frac{w}{32cd^2}$ or $5542 \times \frac{2}{300} = 36''{\cdot}95 = t$, the time of reducing the 300 velocity to 100, or of passing over the space 6108 feet.

90. And, reversing, to find the velocity v, answering to any given time t: Since $t = \frac{w}{32cd^2} \times (\frac{1}{v} - \frac{1}{\mathrm{v}}) = 5542 \times (\frac{1}{v} - \frac{1}{\mathrm{v}})$, therefore $v = \frac{5542\mathrm{v}}{5542 + t\mathrm{v}}$. Here, if t be given $= 30''$, and $\mathrm{v} = 300$; then $v = \frac{5542\mathrm{v}}{5542 + 9000} = \frac{5542}{14542} \times 300 = 114$, the velocity sought.

91. *Corol.* The same form of theorem, $x = \frac{w}{32cd^2} \times$ h. l. $\frac{\mathrm{v}}{v}$, as above is brought out for small velocities, will also serve for the higher ones, if we employ the medium resistance between the two proposed velocities, as was done in prob. 5. Thus, as in the first example of this problem, where the two velocities are 1780 and 1500, the resistance due to the velocity 1700, in the table of resistances, being 76·78, say as $1700^2 : 1780^2 :: 76{\cdot}78 : 84{\cdot}18$, the resistance due to the velocity 1780; then the mean between 84·18 and 59·20, due to 1500 velocity, is 71·69, or rather take 72. Again, as $\sqrt{67{\cdot}93} : \sqrt{72} :: 1600 : 1647$, the velocity due to the medium resistance 72. Hence, as in prob. 5, as $1647^2 : v^2 :: 72 : {\cdot}00002654v^2 =$ suppose av^2, the resistance due to any velocity v, between 1780 and 1500, for the $1\frac{1}{2}$lb ball.

And, as $2^2 : 5{\cdot}546^2 : : av^2 : 7{\cdot}69av^2 = {\cdot}0002041v^2 = bv^2$ suppose, the resistance due to the same velocity with the 24lb ball. Therefore $\frac{bv^2}{24} = f$, and $-\dot{vv} = 32f\dot{x} = \frac{4}{3}bv^2\dot{x}$, and $\dot{x} = \frac{-3\dot{v}}{4bv}$, the correct fluent of which is $\frac{3}{4b} \times \text{h. l.} \frac{v}{v} = \frac{3}{4b} \times \text{h. l.} \frac{178}{150} = \frac{3}{4b} \times \text{h. l.} \frac{89}{75} = 3674 \times {\cdot}171148 = 629$ the velocity sought, nearly the same as before.

PROBLEM XII.

To determine the Ranges of Projectiles in the Air.

92. To determine, by theory, the trajectory a projectile describes in the air, is one of the most difficult problems in the whole course of dynamics, even when assisted by all the experiments that have hitherto been made on this branch of physics; and must still be accounted such, in the full extent of the problem. Even the solutions of Newton, of Bernoulli, of Euler, of Borda, &c, &c, after the most elaborate investigations, assisted by all the resources of the modern analysis, amount to no more than distant approximations, that are rendered nearly useless, even to the speculative philosopher, from the assumption of a very erroneous law of resistance in the air, and much more so to the practical artillerist, both on that account, and from the very intricate process of calculation, which is quite inapplicable to actual service. The solution of this problem requires, as an indispensable datum, the perfect determination by experiment of the nature and laws of the air's resistance at different altitudes, to balls of different sizes and densities, moving with all the usual degrees of celerity. Unfortunately however, hardly any experiments of this kind have been made, excepting those which have been printed in the preceding Tracts of these volumes; which, though so extensive and varied, I cannot yet undertake to pronounce that they are fully adequate to the purpose of all the demands, either of the practical artillerist, or the speculative philosopher.

93. We have indeed, in this extensive course of experiments, completely ascertained and proved many of the important circumstances relating to military projectiles, that were before unknown, or only guessed at. Such as, the nature and force of fired gunpowder; the velocity it will communicate to given bodies; the laws of its action in guns of different lengths, and in charges of different quantities; the effects of the air's resistance on projectiles, discharged with given velocities; the quantity of that resistance, to every degree of velocity, and to every magnitude of surface. In consequence of these, we have deduced the general law of the resistance, in formulæ or functions of the body's velocity; the motion lost in a given time, or in moving through a given space of air; the time in passing through a given space, or the space passed over in a given time; the whole altitude and time of vertical ascent; the terminal or greatest velocity acquired by descending in the air; with other curious particulars.

94. The grand problem of all however still remains, the determination of a projectile's trajectory, when discharged in the air with any velocity, and in any oblique direction. The solutions of this problem that have been accomplished, or rather attempted, by the eminent authors above mentioned, have amounted to nothing more than partial and barren speculations, of no use in real practice, though attempted by talents of the very first order, and with the aid of all the resources of the modern analysis, and even on the most simple hypothesis of the air's resistance, viz, that of the square of the velocity only. But when it is considered, that even this hypothesis is quite inadmissible, and that there is no such thing as any one single power of the velocity, by which it is possible to express the resistance; but that two different powers, at least, are absolutely necessary for that purpose, as we have shown in the foregoing problems, founded on so many and such correct experiments; when these circumstances are considered, it will fully appear that no hope remains of the practicability, if even of the possibility, of ob-

taining a correct and scientific solution of the general problem in that way, or indeed in any way without the aid of some further experiments of another kind, viz, to obtain by accurate experiments, the horizontal ranges and times of flight, of at least one size of ball or shell, discharged at all convenient degrees of elevation, and with all practicable degrees of velocity. If the same could be accomplished with several sizes of shot, it would render the conclusions the more accurate and satisfactory; but those data for one size at least is absolutely necessary; and may be made to answer the purpose of all, by means of calculations founded on certain known scientific principles; for, the ranges and times for other sizes of shot, of equal density, may be ascertained from the experimented one, since all balls discharged with such velocities as shall make their resistance proportional to their momentum, must describe similar curves, at the same elevation of the piece, and the several lines of the trajectories will all be proportional to each other, and to the diameters of the balls.

95. All that can be here done then, in the solution of the present problem, is to collect together some of the best practical rules, founded partly on theory, and partly on practice. In the first place then, it may be remarked, that the initial or first velocity of a ball may be computed from the table of experimented ranges and velocities as in the following problem.

PROBLEM XIII.

To assign the Charge of Powder necessary to project a given Ball with a given Velocity.

96. By the foregoing experiments it has been found that the velocities to different balls, and different charges of powder, with similar guns, are as the square-roots of the weights of the powder directly, and as the square-roots of the weights of the balls inversely. Thus, if it be enquired, with what velocity a 24lb ball will be discharged by 8lb of powder:

it appears in the table, that 8 ounces of powder discharge the 1lb ball with 1600 feet velocity; and because 8lb are = 128 ounces; therefore by the rule, as $\sqrt{\frac{1}{8}} : \sqrt{\frac{128}{24}} :: 1600 : 1600\sqrt{\frac{16}{24}} = 1600\sqrt{\frac{2}{3}} = 1306$, the velocity sought.

97. *Otherwise.* Say, as the root of the weight of the shot, is to the root of double the weight of the powder, so is 1600 to the required velocity. That is, in the present example, as $\sqrt{24} : \sqrt{16} :: 1600 : 1306$, the same velocity.

98. But when the charges bear the same ratio to one another as the weight of the balls, that is when the pieces are said to be alike charged, then the velocities will be equal nearly. Thus, the 1lb ball by the 2 oz charge, being the 8th part of the weight, and the 24lb ball, with 3lb of powder, its 8th part also, will have the same velocity, viz, 800 feet. In like manner, the 1200 tabular velocity, answering to 4 oz of powder, the 4th part of the ball, will nearly belong to the 24lb ball with 6lb of powder, being its 4th part, and the tabular velocity 1600, answering to the 8 oz charge, which is $\frac{1}{3}$ the weight of ball, will equally belong to the 24lb ball with 12lb of powder, being also the $\frac{1}{2}$ of its weight.

99. *Corollary.* In the same manner has been computed the annexed table of the initial velocities, of any cast iron balls, as projected from the usual length of guns, by all the charges of powder, from $\frac{1}{20}$ the weight of the shot, up to the full weight of the same; the strength of the powder being supposed uniform, and of the best kind used by government, or such as will show 40 or upwards, on the line of chords of my powder eprouvette, by the explosion of 2 ounces of the powder.

Proportion of ball to powder, viz. $\frac{b}{c}=$	Veloc. of ball in feet, viz. $1600\sqrt{\frac{2c}{b}}$
20	506
19	519
18	533
17	549
16	566
15	584
14	605
13	628
12	653
11	682
10	716
9	755
8	800
7	855
6	924
5	1012
4	1131
3	1306
2	1600
1	2263

The first column of the table shows how often the wt. or charge of powder c, is contained in the weight of the ball b, being the quotient $\frac{b}{c}$ of the ball divided by the charge of powder, or the ratio of the former to the latter. And the latter column contains the correspondent velocities given to the ball by those charges, being the several values of the theorem $1600\sqrt{\frac{2c}{b}}$ in feet.

100. This table will be useful on many occasions, the velocity being taken out by inspection: So, if it be desired to know what velocity will be given to a ball by $\frac{1}{4}$ of its weight of powder, the table shows between 1100 and 1200 feet. Or, reversewise, it shows what proportion of charge will give the ball any proposed velocity. For numbers intermediate to those in the table, proportion or an allowance may be made; so, if it be desired to know what charge will give any ball a velocity of 1500 feet per second, the table shows that the requisite charge of powder must be nearly a medium between $\frac{1}{3}$ and $\frac{1}{5}$ the weight of the ball. But if the exact quantity of charge be desired, it will be easily obtained by using the general theorem $1600\sqrt{\frac{2c}{b}}$, viz, in the

present case, $1600\sqrt{\frac{2c}{b}} = 1500$, or $16\sqrt{\frac{2c}{b}} = 15$; hence $\sqrt{\frac{2c}{b}} = \frac{15}{16}$, and $\frac{2c}{b} = \frac{15^2}{16^2} = \frac{225}{256}$, or $\frac{c}{b} = \frac{225}{512} = \frac{1}{2\cdot28}$ or $\frac{4}{9}$ nearly; that is, the charge of powder must be almost $\frac{4}{9}$ of the ball's weight, to give it 1500 feet velocity. So, if the ball be an 18-pounder, the charge will be 8lb nearly. But, for a 24-pounder, the charge will be almost $10\frac{2}{3}$lb.

PROBLEM XIV.

To assign the Elevation necessary to give a Piece, discharged with a given Velocity, so as to hit an Object placed at a given Distance.

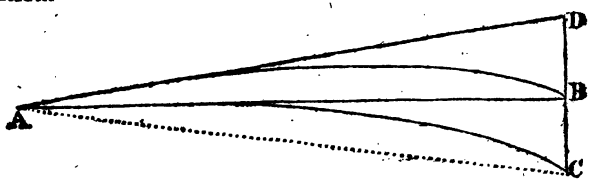
101. This is a very useful problem in the practice of artillery. For, it is obvious, and a circumstance well known, that to hit an object at some distance from the piece, it is necessary to point the piece rather above the visual line of the object, on account of the descent of the shot during the time of its flight, by its natural gravity or weight. For, were the shot discharged exactly in the line of the object, it would be sure, in consequence of its natural descent, to fall too low, and miss the mark. The practice is therefore to elevate and point the piece to a place as much above the object as the ball is supposed to drop in the course of its flight, so that it may just meet the object when it has arrived at the known distance. The quantity of this elevation however is usually a very vague and uncertain degree, being taken without any kind of rule, and only guessed at, more or less, according to the various notions of every practitioner. It is obvious then that this way must at the best afford but a very uncertain result, even when the shot is supposed to be projected exactly in the line the piece is pointed in, which however is commonly far from being the case, on account of several causes of irregularity in the usual practice.

102. One of these causes is, the quantity of windage, by the balls being mostly too small for the bore of the piece:

the consequence of this is, that the ball, in passing through the bore, is kicked about from side to side, which renders it a matter altogether of chance what course the shot may take when out of the piece, as it depends entirely on the part of the bore the shot last strikes, before it issue out of it, whether it may take a course to the right or the left of the plane of direction, or whether it shall rise above the line of direction, or sink below it; as the ball usually takes a course opposite to the last point that it strikes and rebounds from. To go regular and straight along the bore therefore, it is desirable and necessary, that the shot be as large as it can be, conveniently to enter the bore of the piece. This circumstance will also be attended with other considerable advantages, the sparing of gunpowder, as much less of it will suffice, and the shot will have much more velocity, with far less expense of powder.

103. Another disadvantage in the use of artillery, is the shape of the shot, being globular. In consequence of this shape it is, that it is knocked about in the bore, from side to side, and it acquires mostly a rotatory motion round an axis, the effect of which is, after leaving the piece, to cause the ball to deviate and turn aside, more and more in its flight, from the line of direction. Whereas if, instead of the round ball, the shot were made in the shape of a cylinder, more or less long, but of equal diameter with the bore very nearly, the shot would go straight forward along the bore, and out of the piece, at the direction this is pointed in, and arrive at the object with far less cause of deviation. Not only so, but the guns would be much less injured, and the expense of powder also less. Other advantages would also attend this change in the shot; as, from its superior weight, it would range much farther; and, in the sea service, would have not only the common advantage of a heavier shot, but also, from its irregular shape, making a perforation in the side of a ship which could not easily be closed or repaired. By these means the practice could be rendered both more efficacious and more certain; and consequently, instead of merely guess-

ing at the requisite quantity of elevation in the piece, to hit a distant object, the practitioner might be supplied with certain easy and practical rules, for making that allowance of elevation.



104. Thus then, suppose the mark or object B, at the horizontal distance AB, from the gun at A. Now suppose, first, that the ball is discharged in the horizontal direction AB, in which case the gun is said to be laid, and the shot fired point-blank, and is supposed to proceed a considerable way in that direction; yet the ball constantly descends from the horizontal line from the very beginning of the motion, though for a considerable space the descent is so little as to be hardly perceptible. For, in fact, as soon as ever the shot issues from the piece, it is deflected below its direction by the force of gravity, and that more and more rapidly, the longer it is in motion, in proportion as the square of the time. So that the point-blank shot is to be considered only so far as till the descent from the horizontal line, or the angle BAC can be practically observed; or, while that angle is so small, by the curve line AC deviating so little from the right line AB, that the difference may be neglected, without any considerable error. But in cases in which the space and time of flight are so considerable as to deflect the shot in the curve line AC, to the depth of BC below the intended object B, then in practice the gun must be pointed in the direction AD, at a mark D, as much above A as C is below B; for then taking the curved path AB, it may arrive at the object B. We are therefore to consider the projectile in the direction of the right line AD, and find how far it will sink below that line at the point D; which will be very easy, if the time of moving from A to D or B be known, for the line DB will be described by the ac-

tion of gravity, at the rate of 16 feet in the first second, and more or less in proportion to the square of the time. Thus then,

105. Knowing the distance of the object to be struck, at least nearly, compute the velocity that will be given to the shot by the proposed charge of powder, or the charge that will be adequate to a proposed velocity. Thus having the initial velocity, compute by prob. 11, the velocity that will be lost by the shot in passing through the given space of air, between the gun and the object to be struck; which, by subtraction, will give the velocity of the shot when it strikes the object. Take then the mean between this and the initial velocity, which may be accounted a medium uniform velocity with which the shot flies through the air, from the gun to the object. Divide then the given space or distance by that mean velocity, both in feet, and the quotient will be the time of the flight in seconds very nearly. Or find the time at once, by the 11th prob. of passing over the given distance. Lastly, find the space through which a body will freely fall by gravity in that time, and it will be the required quantity nearly, which the piece must be pointed above the object, in order to hit it. Or compute, by prob. 10, the space the body will descend in the air, and that will be the requisite height to be pointed above the object, nearly.

106. *For example.* Suppose the distance of the object be 1000 feet, and the piece a 24-pounder, charged with 6lb of powder. Then, by the last prob. as $\sqrt{24} : \sqrt{12} :: 1600 : 1600\sqrt{\frac{1}{2}} = 1131 = v$ the initial velocity. Next, by prob. 11, $1131 - 231 = 900$, and $N = 1{\cdot}3635$, then $\frac{900}{N} + 231 = 660 + 231 \times 891 = v$ the last velocity at the place of the object; and the medium between 1131 and 891 is 1011; then $\frac{1000}{1011} = 1$ second nearly, the time of flight. But it is well known, that the space freely descended in 1 second is 16 feet. Therefore, if the gun be pointed at a place 16 feet above the object, it will hit it. Or, if 16 be divided by the distance

1000, and the quotient ·016 be sought in a table of natural tangents, it will give 55′, or almost 1°, for the angle the gun must be elevated to. Or, if the log. of ·016 or $\frac{16}{1000}$ be found in the log. tangents, it will give the same angle of elevation 55′.

107. Or, to find the time by the formula in the 11th problem, it will be $32\frac{1}{3} \times$ log of $\frac{V-231}{v-231} \cdot \frac{v}{V} = 32\frac{1}{3} \times$ log. $\frac{1131-231}{891-231} \cdot \frac{891}{1131} = 32\frac{1}{3} \times$ log. $\frac{900.891}{660.1131} = 32\frac{1}{3} \times$ log. $\frac{30\,891}{22.1131} = 32\frac{1}{3} \times$ log. $\frac{4455}{4147} = 32\frac{1}{3} \times ·031114 = 1·003426$, or 1 second nearly, the same as above found, by dividing the distance by the medium velocity.

108. *Example.* 2. To determine the elevation of a 12-pounder gun, charged with 4lb of powder, to hit an object at 500 yards or 1500 feet distance.—Here, the weight of powder being $\frac{1}{3}$ of that of the ball, by the table in prob. 13, the initial velocity is $1306 = V$, the ball's diameter $d = 4·403$, the object's distance $D = 1500 = 1338d \times$ log. $\frac{V-q}{v-q} = 5891 \times$ log. $\frac{1075}{v-231}$ by prob. 11; hence $\frac{D}{1338d} = \frac{1500}{5891} = ·25462$, the number to which log. is $1·7973 = N$, that is, $N = \frac{1075}{v-231}$; this gives $v = \frac{1075}{N} + 231 = 829 = v$ the last velocity at the distance (1500) of the object. Then $\frac{V+v}{2} = 1068$ the middle velocity, and $\frac{1500}{1068} = 1·405''$ the time of flight by this method.

But, by the method in prob. 11, $t = \frac{1338d}{231} \times$ log. $\frac{V-q}{v-q} \cdot \frac{v}{V} = \frac{5891}{231} \times$ log. $\frac{1306-231}{829-231} \cdot \frac{829}{1306} = 25\frac{1}{2} \times$ log. $\frac{1075.829}{598.1306} = 25\frac{1}{2} \times ·057319 = 1·4616''$ the time, more exact, but not much different from the former.

Then, as $1^2 : 1·46^2 :: 16 : 34·1$ feet, the height to be pointed above the object. Or, $\frac{34}{1500} = ·022737$, the tangent of 1° 18′, the angle of elevation to hit the object.

109. *Corollary.* From the preceding calculation, the following general rule is easily derived. D denoting the given distance of the object, in feet; d the diameter of the ball, in inches, obtained from the table of weights and diameters in prob. 10; b the weight of the ball, and c that of the charge of powder, both in pounds; $\text{V} = 1600\sqrt{\frac{2c}{b}}$ the projectile velocity, as given in prob. 13; v the last velocity with which the ball strikes the object; and t the time of the ball's flight. Then,

Divide D by $1338d$, considering the quotient $\frac{\text{D}}{1338d}$ as a log.

Take $\text{N} =$ the natural number of the log. $\frac{\text{D}}{1338d}$.

Take $v = \frac{\text{V}-q}{\text{N}} + q$ the final velocity.

And $t = \frac{1338d}{q} \times$ log. of $(\frac{\text{V}-q}{v-q} \cdot \frac{v}{\text{V}})$ by prob. 11.

Or $t = \frac{2\text{D}}{\text{V}+v}$, an approximation near enough.

Then, $16t^2 = \frac{4\text{D}^2}{(\text{V}+v)^2}$ is the height above the object to be pointed.

Or $\frac{16t^2}{\text{D}} = \frac{4\text{D}}{(\text{V}+v)^2}$ is the tangent of the angle of elevation.

110. So that, the height of the mark to be pointed at, above the object, is nearly as the square of the distance, and the angle of elevation simply as the distance, the projectile velocity being the same. But, in the case of different velocities, the height and the angle will be reciprocally as the square of the velocity nearly. When once the proper angle has been found, and verified by experience, then a dispert may be so fixed on the gun, that the visual line may make with the axis of the piece the proper angle. Now, as such small elevations are hardly perceptible, the reason is plain, why it is thought that the ball proceeds a considerable way from the gun in a straight line. But, small as the curvature is, when the gun is pointed horizontally, it is still smaller when elevated in an angle above the horizon; because, in this case, the gravity acts obliquely on the path, or only

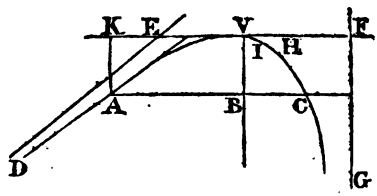
part of it acts on the track, in order to bend it: whereas, in the horizontal or point-blank direction, the whole gravity is employed to that purpose. This diminution will be nearly as the cosine of the angle of elevation, being less and less as that angle is greater, or is pointed nearer the perpendicular; so as that when pointed quite upright, it is not incurvated at all, but the ball ascends perpendicularly upright.

PROBLEM XV.

To determine the Angle of Elevation of a Piece, to produce the greatest Range, also the Extent of that Range, with a given projectile Velocity.

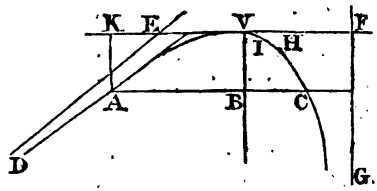
111. In the parabolic theory of projectiles, or those made in empty or unresisting space, it is well known that, for the greatest range, the elevation or direction of the piece, must be such as to bisect the angle contained between the vertex and the plane of the range; that is, when the range is on the horizontal plane, the direction of the piece makes half a right angle with the horizon, or with the vertical line; but when the range is on an ascending plane, or one making an angle above the horizon, then the direction of the piece must make with the vertical line, an angle less than half a right angle, by half the elevation of the plane above the horizon; and when the range is on a descending plane, or one making an angle below the horizon, then the direction of the piece must make with the vertical line, an angle greater than half a right angle, by half the angle of depression of the plane below the horizon. Also, in that theory, the trajectory, or path of the projectile, is a perfect parabola, having the two parts of the curve, on both sides of the axis, quite similar and equal to each other.

112. But, in projects made in the air, none of those beautiful properties take place; on the contrary, every thing is irregular and perplexed; the ascending and descending



branches AV and VC, of the path, on the right and left of the vertex, being quite unequal and dissimilar; and, instead of being parabolic curves, they are rather different legs of an hyperbolic kind, having dissimilar asymptotes, DE and FG, the former inclined to the horizontal line EF, and the latter perpendicular to it.

113. In the parabolic theory, the horizontal velocity of the shot, or its motion estimated in the horizontal direction, is always constant and uniform; because the action of gravity, being the only force that constantly acts on it, is exerted in a direction perpendicular to the horizon. But it is quite different in the other case; where, besides the force of gravity acting always perpendicular to the horizon, there is the resistance of the air, acting in the line of the ball's path, or directly opposite to its motion, which both diminishes its ascent, or descent, and its horizontal motion; on which account, its horizontal velocity must be continually diminishing, in the whole course of the ball's motion, both in the ascending and descending branch of its path. Hence too it will happen that, in ascending, the ball's motion will decrease very fast, from these double causes, and it will not ascend near so high as in the former case, or the vertex of the curve will not be either so high, or so far distant in the horizontal direction; that is, AB and VB will both of them be much smaller than they would have been in the former case. Again, in the descending branch VC, after having passed the vertex V, the gravity accelerates its motion downwards, the horizontal motion continuing still to decrease; by which means it happens, that the direction of the descent approaches always nearer and nearer to the perpendicular direction, or towards a parallel with VB; till at length, the horizontal motion being quite destroyed, the ball descends by its gravity quite perpendicularly downwards, with a velocity which, though continually increasing, can never exceed, nor



indeed quite equal, its terminal velocity, such as is peculiar to the particular size and weight of the ball, as assigned in prob. 10, which therefore is the utmost limit of the velocity that the shot can descend with.

114. From this view of the ball's motion, it is manifest that the two branches of the path are both of them of the hyperbolic kind, having an asymptote, or tangent at an infinite distance, in a particular situation, being oblique, as DE, in the ascending branch, but perpendicular, as FG, in the descending. Further, the velocity in the ascending branch is quicker, but the time in motion less, than in the descending branch, as the motion in the latter can only approach a certain limit, called the terminal velocity. The ascending branch too is less curved than the descending one.

115. Again, the slowest motion of the shot, in the parabolic theory, is at the vertex; but in the air, or resistance theory, the point of slowest motion is a little beyond the vertex, in the descending branch, as at the point H; because the motion there, nearly horizontal, is much resisted by the medium, while it is but little assisted by the perpendicular descent of gravity. For the same reason too, the point of greatest curvature will not be at the vertex, as in the parabolic theory, but a little way beyond it, as at I. Further, the ordinate BC in the descending branch, is much less than that AB in the ascending one, and both of them far less than in the parabola, sometimes as much as in the proportion of 10 or 20 to 1; the altitude BV is also much less.

116. Lastly, the angle of elevation of the piece, to yield the greatest range or amplitude AC, with the same velocity, is not when it is equal to half the angle of the plane AC with the vertical line AK, as in the parabola; but, on the contrary, is considerably less than half, and the more so both as the shot is smaller, and the velocity is greater; because, in the

lower pitch, the shot has much less quantity of the resisting air to pass through, than in the higher ones. On these accounts the angle for the greatest horizontal range, may be at all degrees between 45° and 30°, the slowest motions and largest shot being almost at 45°, but gradually more and more below that degree, as the shot is smaller and the velocity greater, till at length, with the most rapid motions and the smallest shot, the angle is little above 30°.

117. From all these circumstances it partly appears, how difficult a thing it must be to institute a calculation that may produce, with any tolerable degree of practical utility, all, or indeed any of the circumstances of a projectile's path. But when it is further considered that the formula by which the resistance is expressed, in terms of the velocity, is necessarily rather complex, consisting of two different powers of the velocity at least, the attempt will soon evince that the accomplishment of the required object, is a thing on the very verge of impossibility. Hence, all that has been ever done in this matter, by the ablest mathematicians that ever existed, with the aid of all the resources of the most refined methods of analysis, has amounted to nothing more than imperfect and rather distant approximations; nor is much more now to be accomplished, or expected, on the circumstances of the path, since the correct and general expression of the resistance has been obtained; and so the case must continue to remain, till many and accurate experiments can be obtained, on the ranges and times of flight of projectiles, with all degrees of elevation and velocity; but which, it is to be feared, are not likely soon to be expected. The approximations on this head, that have been made by Newton, and Robins, and Euler, and professor Robison, appear to approach nearest to practical usefulness, and of some of these we shall endeavour to avail ourselves on this point; making however several necessary alterations and corrections in them, from the more correct nature and quantity of the resistance which we have been enabled to derive from our more correct and extensive experiments.

118. And first then, having given the initial velocity, and the size and nature of the shot, we may thence find nearly the elevation of the piece to produce the greatest amplitude, and the extent of that range.

Now these will be found nearly by means of the following table, altered from professor Robison's, in the Encyclopædia Britannica, and founded on an approximation of Sir I. Newton's. The numbers in the first column, multiplied by the terminal velocity of the ball, give the initial velocity; and the numbers in the last column, being multiplied by the height, give the greatest ranges; the middle column showing the elevations to produce those ranges.

Table of Elevations giving the greatest Range.

Initial vel. div. by v.	Elevation.	Range div. by a.
0·6910	44° 0′	0·4110
0·9445	43 15	0·6148
1·1980	42 30	0·8176
1·4515	41 45	1·0210
1·7050	41 0	1·2244
1·9585	40 15	1·4278
2·2120	39 30	1·6312
2·4655	38 45	1·8346
2·7190	38 0	2·0379
2·9725	37 15	2·2413
3·2260	36 30	2·4447
3·4795	35 45	2·6481
3·7330	35 0	2·8515
3·9865	34 15	3·0549
4·2400	33 30	3·2583
4·4935	32 45	3·4616
4·7470	32 0	3·6650
5·0000	31 15	3·8684

119. To use this table then, divide the given initial velocity by the terminal velocity peculiar to the ball, found in the table in prob. 10, and look for the quotient in the first column here annexed. Against this, in the 2d column will be found the elevation to give the greatest range; and the number in the 3d column multiplied by a, the altitude due to the terminal velocity, also found in the table in problem 10, will give the range, nearly.

120. *Ex.* 1. Let it be required to find the greatest range of a 24lb ball, when discharged with 1640 feet velocity, and the corresponding angle to produce that range. By the table in prob. 10, the terminal velocity of the 24lb ball is 419, and its producing altitude 2729: hence $\frac{1640}{419} = 3·92$, nearly

equal to 3·9865 in the 1st column of the table, to which corresponds the angle 34° 15′, being the elevation to produce the greatest range; and the corresponding number 3·0549, in the 3d column, multiplied by 2729, gives 8336 feet, for the greatest range, being rather more than a mile and a half.

121. *Exam.* 2. In like manner, the same ball discharged with the velocity 860 feet, will have for its greatest range 4081 feet, or above ¾ of a mile, and the elevation producing it 40°.

122. *Corollary.* These examples, and indeed the whole table in the 10th problem, are only adapted to the use of cannon balls. But it is not usual, and indeed not easily practicable, to discharge cannon shot at such elevations, in the British service, that practice being the peculiar office of mortar shells. On this account then it will be necessary to make out a table of terminal velocities, and altitudes due to them, for the different sizes of such shells. The several kinds of these in present use, are denominated from the diameters of their mortar bores in inches, being the five following, viz, the 4·6, the 5·8, the 8, the 10, and the 13 inch mortars, as in the first column of the following table. But the outer diameters of the shells are somewhat smaller, to leave a little room or space for windage, as contained in the 2d column.

Table of dimensions, &c, of Mortar Shells.

Diam. of mortar	Diam. of shells	Wt. of shells empty	Wt. of shells fielld	Weight of equal solid	Ratio of shell to solid	Terminal velocity	Alt. *a* due to veloc.
inches	inches	lbs	lbs	lbs		feet	feet
4·6	4·53	8·3	9	12¾	1·42	318	1580
5·8	5·72	16·7	18	25½	1·42	356	1980
8	7·90	43·8	47	67	1·42	420	2756
10	9·84	85·5	91¼	130	1·42	468	3422
13	12·80	187·8	201	286	1·42	534	4430

The 3d column contains the weight of each shell when

empty; the 4th when the hollow part is filled with powder: the diameter of the hollow is usually $\frac{7}{10}$ of that of the mortar. On account of the vacuity of the shell being filled only with gunpowder, the weight of the whole so filled, and contained in column 4, is much less than the weight of the same size of solid iron, and the corresponding weights of such equal solid balls are contained in col. 5. The ratio of these weights, or the latter divided by the former, occupies the 6th column.

123. Now, because the loaded or filled shells are of less specific gravity, or less heavy, than the equal solid iron balls, in the ratio of 1 to 1·42, as in column 6, the former will have less power or force to oppose the resistance of the air, in that same proportion, and the terminal or greatest velocity, as determined in the 10th problem, will be correspondently less. Therefore, instead of the rule there given, viz, $178\sqrt{d}$, for that velocity, the rule must now be $178\sqrt{\frac{d}{1{\cdot}42}} = 149{\cdot}4\sqrt{d} = v$, the diameter of the shell being d; that is, the terminal velocities will be all less in the ratio of 149·4 to 178. Now, computing these several velocities by this rule, to all the different diameters, they are found as placed in the 7th column; and in the 8th or last column are set the altitudes which would produce these velocities in vacuo, as computed from this theorem $\frac{vv}{64}$.

124. Having now obtained these terminal velocities, and their producing altitudes, for the shells, we can, from them and the former table of ranges and elevations, easily compute the greatest range, and the corresponding angle of elevation, for any mortar and shell, in the same way as was done for the balls in this problem. Thus, for example, to find the greatest range and elevation, for the 13 inch shell, when projected with the velocity of 2000 feet per second, being nearly the greatest velocity that shells can be discharged with. Now, by the method before used, $\frac{2000}{534} = 3{\cdot}746$; opposite to this, found in the first column of the table of ranges, corre-

sponds 35° 0′ for the elevation in the 2d column, and the number 2·8515 in the 3d column; this multiplied by the altitude 4430, gives 12632 feet, or almost $2\frac{1}{2}$ miles for the greatest range.

124. This however is much short of the distance which, it is said, the French have lately thrown some shells at the siege of Cadiz, viz, 3 miles, or more; which it seems has been effected by means of a peculiar piece of ordnance, and by loading or filling the cavity of the shell with lead, to render it heavier, and thus make it fitter to overcome the resistance of the air. Let us then examine what will be the greatest range of our 13 inch shell, if its usual cavity were quite filled with lead, when discharged with the projectile velocity of 2000 feet.

125. Now the diameter of the cavity, being about $\frac{7}{10}$ of that of the mortar 13, will be nearly 9 inches. And the weight of a globe of lead of this diameter is 139·3lb: which added to 187·8, the weight of the shell empty, gives 327lb, the whole weight of the shell when the cavity is filled with lead, which was found 286 when supposed all of solid iron, their ratio or quotient is ·8783. Then, as before, the theorem will be $178\sqrt{\frac{d}{\cdot 8783}} = 190\sqrt{d}$ for the terminal velocity; which, when $d = 12\cdot 8$, becomes 680 for the terminal velocity; therefore its producing altitude is $\frac{680^2}{64} = 7225$. Then, by the same method as before, $\frac{2000}{680} = 2\cdot 941$; which number found in the first column of the table of ranges, the opposite number in the 2d col. is 37° 20′ for the elevation of the piece, and in the 3d column 2·2153, which multiplied by 7225, gives 16005 feet, or 3 miles and 165 feet.

So that our 13 inch shells, discharged at an elevation of about $37\frac{1}{3}$ degrees, would range nearly the distance mentioned by the French, when filled with lead, if they could be projected with so much as 2000 feet velocity, or upwards. This however it is thought cannot be effected by our mortars; and that it is therefore probable the French, to give

such a velocity to those shells, must have contrived some new kind of large cannon on the occasion. It is said these shells, after being filled with lead, are bored; that they seldom burst; and when they do, the explosion is inconsiderable.

PROBLEM XVI.

To determine the Circumstances of Ranges at other oblique Elevations.

126. Having shown, in the preceding articles and problems, how, from our theory of the air's resistance, can be found, first the initial or projectile velocity of shot and shells; 2dly, the terminal velocity, or the greatest velocity a ball can acquire by descending by its own weight in the air; 3dly, the height a ball will ascend to in the air, being projected vertically with a given velocity, also the time of that ascent; 4thly, the *greatest* horizontal ranges of given shot, projected with a given velocity; as also the particular angle of elevation of the piece, to produce that greatest range. It remains now to enquire, what laws and regulations can be given respecting the ranges, and times of flight, of projects made at other angles of elevation.

127. Rules for the general solution of this problem would be best derived from experiments; and these should be made at all elevations, and with all charges, and with various sizes of balls; observing both the ranges and times of flight, in every experiment. Such experiments would give us the relations existing, in all cases, among all these four terms, viz, the ranges, the times of flight, the velocities or charges, and the size of the balls. Numerous and various as our experiments are, as before related, and fruitful as they are in useful consequences, we have obtained but a small portion of those above alluded to; nor do I know of any proper set of such experiments anywhere to be found. Such must therefore still remain a valuable desideratum. The few that we have been able to make, are those collected in arts. 8 and 24 of

this Tract; and these afford us but very few and imperfect rules, and are chiefly these following. 1st, That the ranges with the one-pound balls, at an elevation of 15 degrees, are nearly proportional to the times of flight. 2nd, That the ranges with the 3-pound balls, at 45° elevation, are nearly as the times of flight, and also as the projectile velocities. Besides these inferences, it does not appear that the experiments are extensive enough to afford many more useful conclusions. By trials however among many of the numbers in the table in art. 24, it appears that in most of them, at an elevation between 45° and 30°, the time of flight is nearly equal to $\frac{1}{4}$ of the square root of the range; in which respect it nearly agrees with the similar rule for the time of flight, in the parabolic theory, at the angle (45°) for the greatest range, which time it is well known is equal to $\frac{1}{4}$ of the square root of the said range in feet, and that whether the shot consist of materials very dense or very light. Whence it is probable that, with the help of a few other ranges at several elevations, some general relations might be evinced between the ranges and the times of flight, with the tangents of the elevations.

But such experiments and enquiries as these, unfortunately, it is no longer in my power either to procure, or by any means to promote. We can therefore only endeavour to render, without them, what service we can, to the state and to philosophy, by such means as are in our power.

128. There are some few theoretical principles which it may be useful to notice here, as first mentioned by professor Robison. Thus, balls of equal density, discharged at the same elevation, with velocities which are proportional to the square-roots of their diameters, will describe similar curves; because then the resistances will be in proportion as the momenta or quantities of motion. For, the resistance r, is d^2v^2 nearly; d being the diameter, and v the velocity. But, v being as $\sqrt{d}$, v^2 will be as d; consequently d^2v^2 will be as d^3, that is, r is as d^3. But the momentum is as the magnitude or mass, which is as d^3 also, the cube of the diameter. Therefore the resistance is proportional to the momentum, when

the velocity is as $\sqrt{d}$, the square-root of the diameter of the ball. In this case then, the horizontal velocity at the vertex of the curve will be proportional to the terminal velocity; also the ranges, and heights, and all the other similar lines in the curve, will be proportional to d, the diameter of the ball. And this principle may be of considerable use: for, by means of a proper series of experiments on one ball, projected with different velocities and elevations, tables may be constructed, by which may be ascertained the motions of all similar cases.

129. The following table also, deduced partly from theory and partly from experiment, may be found of use, by analogy, in many other cases.

Table of the Motions of a 24-pound Shot, projected at 45° Elevation.

Velocity per second.	Range in vacuo.	Range in the air.	Range corrected	Height the ball rises to.
feet	yards	yards	yards	yards
200	415	320	330	100
400	1658	1000	1019	300
600	3731	1391	1419	400
800	6632	1687	1719	464
1000	10362	1840	1878	515
1200	14922	1934	1978	561
1400	20310	2078	2129	606
1600	26528	2206	2264	650
1800	33574	2326	2391	694
2000	41450	2438	2510	738
2200	50155	2542	2622	778
2400	59688	2640	2726	816
2600	70050	2734	2823	852
2800	81241	2827	2916	887
3000	93262	2915	3002	922
3200	106111	2995	3086	996
1	2	3	4	5

This table contains, in the first column, the velocities, in feet, of a 24-pound shot, projected at an angle of 45 de-

grees; the 2nd column shows the distances, in yards, to which the ball would go, in vacuo, on a horizontal plane; the 3rd column contains the distances to which it will range through the air, considered all of the same density as at the earth's surface; the 4th column shows the same ranges corrected for the diminution of the air's density as the ball ascends, being therefore called the corrected range; and the 5th, or last column, shows the height to which the ball rises in the air, or the height of the vertex of the curve above the plane.

130. By this table it appears what an immense difference there is between the motions through the air and in a void, especially with the large velocities. It is seen that the ranges through the air, instead of increasing in the duplicate ratio of the initial velocities, as in unresisting space, really increase slower than those velocities, in all the actual cases of military service; and in the most useful cases, viz, from 800 to 1600 feet, they increase nearly as the square-roots of the velocities. A set of similar tables, made with the same shot, at other elevations, would nearly complete what might be done by theory, as well as useful and necessary for practice.

To use the foregoing Table.

131. Suppose it were required to find the dimensions of the path described by a 12-pound shot, discharged with 1600 feet velocity, and at an elevation of 45 degrees.

Here, recollecting that the curves are similar, and their corresponding lines proportional to the diameters of the shot, when these are discharged with velocities that are to one another as the square-roots of the same diameters; we must therefore first find what velocity of the 24lb, or tabular ball, corresponds to the 1600 velocity of the 12-pounder, in this manner. The diameters of the two balls being 4·403 and 5·546, it will be, as $\sqrt{4{\cdot}403} : \sqrt{5{\cdot}546} :: 1600 : 1796$ the correspondent velocity of the 24-pounder; which being sought in the table, the range and height answering to it, are

there seen to be ·2386 and 692; therefore as 5·546 : 4·403 :: $\left\{\begin{array}{l}2386 : 1894 \text{ yards, the range,}\\ 692 : 549 \text{ yards, the altitude.}\end{array}\right.$

That is, the range is about $1\frac{1}{12}$ mile, and the height almost $\frac{1}{3}$ of a mile.

132. Again, to find the range and greatest altitude of a 9-pound ball, projected with 1500 feet velocity. The diameter of the 9lb ball being 4 inches, we have $\sqrt{4} : \sqrt{5{\cdot}546}$ :: 1500 : 1766; to which in the table correspond 2343 and 680 for the tabular range and altitude. Then

5·546 : 4 :: $\left\{\begin{array}{l}2343 : 1690 \text{ the range, almost 1 mile.}\\ 680 : 491 \text{ the height, almost } \frac{1}{3} \text{ of a mile.}\end{array}\right.$

133. The same table may also serve for computing the ranges of mortar-shells. For this purpose, we have only to find what must be the initial velocity of the 24-pound shot corresponding to the proposed velocity of the shell. This must be deduced from the diameter and weight of the shell, by making the velocity of the 24-pounder such, that the ratio of its weight to the resistance may be the same as in the shell, after the manner of the like accommodation in problem 15.

Thus, to find the range of the 13-inch shell, projected with the velocity of 2000 feet per second, at the 45° elevation.—Here, the diameter being 12·8, we have $\sqrt{12{\cdot}8} : \sqrt{5{\cdot}546}$ :: 2000 : 1317, and as 178 : 149·4 :: 1317 : 1105 the correspondent velocity of the 24-pounder; to which in the table answers the range 1930; then as 5·546 : 12·8 :: 1930 : 4455 yards, or about $2\frac{1}{2}$ miles, the range.

PROBLEM XVII.

To determine the Penetrations of Balls into different Substances.

134. The penetrations of the balls, into the blocks of wood, formed a particular object of enquiry, in the experiments with the ballistic pendulum. When a body in motion

strikes an immoveable obstacle, it either recoils, or penetrates into it, until its whole motion is destroyed by the resistance. Now the recoil takes place when the bodies are endued with elasticity, and will always happen when the elasticity is perfect, that is, when they possess a power of restoring themselves completely to the same shape, which they had before the stroke, after having their forms charged by the said stroke; and a recoil will often happen when the elasticity is imperfect, or only partial, and the velocity and force of the striking body may not be sufficient to overcome the resisting strength of the obstacle, thereby preventing the penetration and lodging of the striking body. When the body and obstacle are quite destitute of this faculty of elasticity and restoring themselves, the body will penetrate into the obstacle till all its motion be destroyed, and remain at rest, as in the case of a body falling down into a mass of soft clay, &c. In both cases the ball will make an impression; in the former case there will be a restoration to the former shape; in the other the impression will remain; and in this consists the difference between elastic and unelastic bodies.

135. But, between these two kinds of bodies, there may be an infinite number of other sorts, endued with different degrees of this power of restoring themselves. Indeed, it may with certainty be affirmed, that there is not in the whole world any body that is perfectly elastic, nor yet quite devoid of elasticity; but that all possess this quality in some degree. However elastic a body may seem to be, yet it will still retain some small mark, where an impression has been made; and no body has yet been discovered which, having received an impression, will not restore itself in some small degree. To explain this proposition, we need only consider the cavity which is made by the collision of the two bodies, and which is always made, whether the body is elastic or not; and, in this view, it is the same thing, whether the cavity remain unaltered, or restore itself, either wholly or in part, to its former condition. And it is clear from experiment, that if a ball be impelled against such an obstacle as a bank of earth,

a rampart, a block of wood, &c, the ball will lodge in it without any sensible recoil; so that, in the present consideration, if the body be elastic, its elasticity may be neglected.

136. When a ball is discharged against such an obstacle, it will not only make an impression, but will penetrate to a certain depth; and as this cannot happen without the ball suffering a great resistance from the obstacle, it will have its motion gradually diminished, and finally quite destroyed. To find the depth to which the ball can penetrate, we must determine the resistance it suffers while it penetrates into the obstacle; for, whatever matter the obstacle consists of, whether wood, or earth, &c, a greater cavity will always require a greater force; and since the elasticity is not considerable, the ball will meet with the same resistance nearly, after entering a certain depth, as at the beginning of its penetration; excepting that, in such cases, the farther the penetration is continued, the resistance will be rather increased by the greater accumulation and condensation of the parts of the obstacle in the front of the striking body. The resistance is therefore nearly a constant and uniform force, no way depending on the velocity; and so will be nearly similar to the action of gravity, by which it happens that a body, projected directly upwards, loses equal quantities of its motion in equal times, whether the velocity be quick or slow. The quantity of this force depends on the strength of the obstacle, and the width of the cavity made by the ball, which is proportional to the square of the ball's diameter. Hence the calculations for the circumstances of this motion and resistance, considering this last as a constant quantity, will be similar to the resisting force of gravity on a body projected upwards, or to any motion whatever resisted by a force that is constant.

137. Therefore, let f denote the constant resisting force, or strength of the wood, or other matter; s the space or depth penetrated; v the first velocity, and t the time of penetration; also $g = 16\frac{1}{12}$ feet, or 16 feet only, the space a

body naturally descends by gravity in the first second of time, or half the velocity generated or destroyed in 1 second. Then, by the principles of constant forces, $f = \frac{v^2}{4gs}$, and $t = \frac{2s}{v}$.

138. Now, for an example, I have found, by a medium of several experiments, that a cast-iron ball, of 1·96 inches diameter, discharged into the end of a block of elm wood, or in the direction of the fibres, with a velocity of 1600 feet per second of time, penetrated 20 inches deep into its substance. It is proposed then to determine the time of the penetration; also the strength or resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.

Here we have $v = 1600$ feet, and $s = 20$ inches $= \frac{5}{3}$ feet: therefore $t = \frac{2s}{v} = \frac{10}{3 \times 1600} = \frac{1}{480}$, the time, or part of a second, during the whole penetration. Also $f = \frac{v^2}{4gs} = \frac{1600^2}{4 \times 16 \times \frac{20}{12}} = 24000$. That is, the resisting force of the wood, in this instance, is to the force of gravity, as 24000 to 1.

139. *Exam.* 2. In another instance the same ball, with 1200 feet velocity, penetrated 15 inches.—Here then $f = \frac{v^2}{64s} = \frac{1200^2}{64 \cdot \frac{15}{12}} = 18000$, the strength of the wood. And $t = \frac{2s}{v} = \frac{\frac{30}{12}}{1200} = \frac{1}{480}$, the time, the same as before.

140. *Exam.* 3. In a third instance, the 3-pound ball, of 2·78 inches diameter, with 1500 feet velocity, penetrated 30 inches deep.—Here then

$f = \frac{1500^2}{64 \cdot \frac{30}{12}} = 14062$, the strength of the wood. And

$t = \frac{\frac{30}{12}}{1500} = \frac{5}{1500} = \frac{1}{300}$, the time in this case.

141. *Exam.* 4. In a 4th case, the last ball, with 1060 velocity, penetrated 16 inches deep.—Hence

$f = \frac{1060^2}{64 \cdot \frac{16}{12}} = 13200$, the strength in this case.—And

$t = \frac{\frac{16}{12}}{1060} = \frac{1}{898}$, the time in this case.

142. In these four examples, the results are various, both for the strength of the wood, and the time of penetration, though the data, from which they are computed, are the mediums of several experiments; doubtless, the consequence of several unavoidable circumstances, as, the irregularity among the different experiments, and the different blocks of the wood, and the resistance of the wood not being exactly a constant quantity, owing to the condensation and elasticity of the parts of it driven in before the ball. In this uncertainty, we may take a medium among all the four results, both for the strength of the timber, and the time of penetration, as follows.

Values of f.	*of t.*
24000	480
18000	480
14062	300
13200	398
4) 69262	4) 1658
$f =$ 17315 mediums.	415 . . . $t = \frac{1}{415}$.

So that, the mediums among these 4 cases, give us $\frac{1}{415}$ part of a second, for the time of the ball's penetration; and 17315 for the strength of the timber, or that its resistance is to the force of gravity, as 17315 to 1.

Otherwise.

143. The calculation may be otherwise instituted thus. Putting, as before, d for the ball's diameter, and f for the strength or firmness of the matter in the obstacle; then shall d^2f be proportional to the resistance; which resistance let be expressed by the weight of a column of water of the same diameter d as the ball, and the height f, its specific gravity being 1; and let the specific gravity of the matter of the ball be denoted by n, which in the present case is $7\frac{1}{2}$ nearly; and $a = \cdot 7854$. Then ad^2 will be the area of the section, and $\frac{2}{3}and^3$ the weight of the ball; so that the resistance will be to the weight of the ball, as ad^2f to $\frac{2}{3}and^3$, or as f to $\frac{2}{3}nd$,

that is, as $\frac{3f}{2nd}$ to 1 or gravity. Now, the velocity generated, or destroyed, by gravity, in describing the space s, is $8\sqrt{s}$, thus found, as $\sqrt{16} : 32 : : \sqrt{s} : 8\sqrt{s}$; and, when the space is the same, the velocities generated or destroyed, are as the square-roots of the forces, or the forces as the squares of the velocities; and the velocity destroyed by the force $\frac{3f}{2nd}$, in describing the space s, being v; therefore as $1 : \frac{3f}{2nd} : : 64s : v^2$; hence $\frac{3f}{2nd} = \frac{v^2}{64s}$. Consequently $f = \frac{nd}{3} \times \frac{v^2}{32s}$, and $s = \frac{ndv^2}{96f}$.

Otherwise again.

144. Because the resistance is $\frac{3f}{2nd}$, the gravity being 1, therefore $v\dot{v} = -\frac{3fg}{nd}\dot{s}$, and the fluents give $v^2 = v^2 - \frac{6fgs}{nd}$; and when $v = 0$, or the motion ceases, then $v^2 = \frac{6fgs}{nd}$. Hence $f = \frac{ndv^2}{6gs} = \frac{ndv^2}{96s}$, and $s = \frac{ndv^2}{96f}$, the same as the last.

145. Hence it appears that the depth penetrated, is as the density and diameter of the ball and the square of the velocity, divided by the strength, or resisting force, of the matter of the obstacle. So that, if equal balls be discharged against the obstacle, or block, then shall the depth of the cavities made by the balls be as the squares of their velocities; with which the experiments nearly agree.

146. Hence also it follows, that if unequal balls, but of the same density or matter, be discharged with equal velocity against an obstacle, the depths of the cavities will be proportional to the diameters of the balls: so that, a greater ball will not only make a wider aperture, but will also penetrate farther into the obstacle, than a smaller ball, every other circumstance being the same in both. Again, if it be known, from experiment, what is the depth of the cavity, which a given ball, moving with a known velocity, makes in an obstacle, we can from that find the value of f, which denotes the strength or firmness of the matter of which the obstacle consists: and, in this manner, we may from experiments be

enabled to compare the strength of different kinds of matter with each other, which different bodies consist of. To illustrate these inferences, the following practical examples are here annexed.

147. *Exam.* 1. Taking here again the same example as at art. 138, to compute it by this rule, $\frac{ndv^2}{96s} = \frac{7\frac{1}{2}dv^2}{96s} = \frac{5dv^2}{64s}$ for the value of f; and $\frac{5dv^2}{64f}$ for the value s the space penetrated. Now, in this example we have given $d = 1{\cdot}96 = 2$ nearly, $v = 1600$, and $s = 20$; therefore $\frac{5dv^2}{64s} = \frac{5.2.1600^2}{64 . 20} = \frac{200^2}{2} = 20000 = f$ the value of the elm's resistance by this rule, in the first case.

148. *Exam.* 2. In like manner, for the 2nd example, in art. 139, where $d = 2$, $v = 1200$, $s = 15$, the rule gives $\frac{5 . 2 . 1200^2}{64 . 15} = \frac{2 . 150^2}{3} = \frac{45030}{3} = 15000$ the value of f in this case.

149. *Exam.* 3. Again, for the 3rd example in art. 140, when $d = 2{\cdot}78$ or $2{\cdot}8$ nearly, and $v = 1500$, also $s = 30$; give $\frac{5 . 2{\cdot}8 . 1500^2}{64 . 30} = \frac{1.4 . 1500^2}{8^2 . 3} = 16406$ the value of f in this case.

150. *Exam.* 4. Lastly, for the 4th example, in art. 141, where again $d = 2{\cdot}8$, also $v = 1060$, and $s = 16$; which give $\frac{5 . 28 . 1060^2}{8^2 . 16} = \frac{14.265^2}{8^2} = \frac{7 . 265^2}{32} = 15362$ the value of f in this case.

151. Then to take the medium among these four.

$$\begin{array}{r} 20000 \\ 15000 \\ 16406 \\ 15362 \\ \hline 4\,)\,66768 \end{array}$$

the medium value 16692 of f, by this rule, being nearly the same as before, viz, 17315, by the first rule.

152. *Exam.* 5. To find how far a 24-pound ball of cast

iron will penetrate into a block of elm, when discharged with a velocity of 1600 feet per second.

Here, because, by art. 139, the ball of 2 inches diameter, with 1200 feet velocity, penetrated 15 inches deep; therefore, by art. 141, the depth for the 24lb ball, or 5·55 inches diameter, with 1600 feet velocity, will be found by this proportion; as $2 \times 1200^2 : 5{\cdot}55 \times 1600^2 : 15 : \frac{5{\cdot}55 \times 4^2 \times 15}{2 \times 3^2} =$ 37 inches nearly, or 3 feet 1 inch, the penetration.

153. *Exam.* 6. It is stated by Mr. Robins (vol. 1, p. 273, of his works), that an 18-pounder ball, discharged with a velocity of 1200 feet per second, penetrated 34 inches into sound dry oak. It is required then to ascertain the comparative strength or firmness of oak and elm.—By art. 140, the strength of the obstacle being as the diameter of the ball and the square of the velocity directly, and the space penetrated inversely; and the diameter of the 18lb ball being 5·04 inches; we shall have,

$\frac{f}{F} = \frac{dv^2s}{Dv^2s} = \frac{2 \times 12^2 \times 34}{5{\cdot}04 \times 12^2 \times 15} = \frac{2 \times 34}{5{\cdot}04 \times 15} = \frac{17}{19}$ nearly.—Whence it would seem, that elm timber resists less than oak, in the ratio of about 17 to 19.

155. *Exam.* 7. A 24-pounder ball being discharged into a bank of firm earth, with a velocity of 1300 feet per second, penetrated 15 feet. It is required then to ascertain the comparative resistances of the earth and elm wood. Comparing the numbers here with those of the 5th example with the same ball, it is $\frac{f}{F} = \frac{v^2s}{V^2s} = \frac{16^2 \times 15 \times 12}{13^2 \times 37} = \frac{256 \times 180}{13^2 \times 37} = \frac{46080}{6253} =$ $7\frac{1}{3}$ nearly. That is, elm timber resists about $7\frac{1}{3}$ times more than the earth.

156. *Exam.* 8. To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; supposing it discharged with a velocity of 1700 feet per second, and that the lead does not change its figure by the stroke against the wood.—Here $D = \frac{3}{4}$, $N = 11\frac{1}{3}$ the specific gravity of lead, and $n = 7\frac{1}{3}$ nearly, that of cast iron. Then, as before $\frac{DNv^2s}{dnv^2}$,

$= \frac{\frac{3}{4} \times 11\frac{1}{3} \times 17^2 \times 20}{2 \times 7\frac{1}{3} \times 16^2} = \frac{17^2 \times 30}{16^2 \times 44} = 13$ inches nearly, the depth of penetration.—But, as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted more than twice as much as mine; or, else, which is much more probable, that the defect in his penetration resulted from the change of figure in his leaden ball, from the blow against the wood; as such bullets have been observed to be quite flatted by such a stroke.

157. *Exam.* 9. A one-pound ball, projected with a velocity of 1200 feet per second, having been found to penetrate 15 inches deep into sound elm: It is proposed to ascertain the time of passing through every single inch of the 15, and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.—Here, the velocity v being 1200 feet, or $1200 \times 12 = 14400$ inches; and the velocities and times being as the square roots of the spaces, in constant retarding forces, as well as in accelerating ones; and t being $= \frac{2s}{v} = \frac{2 \times 15}{12 \times 1200} = \frac{30}{14400} = \frac{1}{480}$ part of a second, the whole time of passing through the 15 inches; therefore as

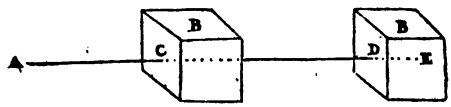
$$\sqrt{15} : \sqrt{15} - \sqrt{14} :: v :$$

Veloc. lost.		Time in the
$\frac{\sqrt{15}-\sqrt{14}}{\sqrt{15}}v = 40{\cdot}7 :: t : \frac{\sqrt{15}-\sqrt{14}}{\sqrt{15}}t =$	·00007	1st inch
$\frac{\sqrt{14}-\sqrt{13}}{\sqrt{15}}v = 42{\cdot}2 :: t : \frac{\sqrt{14}-\sqrt{13}}{\sqrt{15}}t =$	·00007	2nd
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{15}}v = 43{\cdot}8 :: t : \frac{\sqrt{13}-\sqrt{12}}{\sqrt{15}}t =$	·00008	3rd
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{15}}v = 45{\cdot}7 :: t : \frac{\sqrt{12}-\sqrt{11}}{\sqrt{15}}t =$	·00008	4th
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{15}}v = 47{\cdot}8$ &c. $\frac{\sqrt{11}-\sqrt{10}}{\sqrt{15}}t =$	·00008	5th
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{15}}v = 50{\cdot}3$ $\frac{\sqrt{10}-\sqrt{9}}{\sqrt{15}}t =$	·00009	6th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{15}}v = 53{\cdot}2$ $\frac{\sqrt{9}-\sqrt{8}}{\sqrt{15}}t =$	·00009	7th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{15}}v = 56{\cdot}6$ $\frac{\sqrt{8}-\sqrt{7}}{\sqrt{15}}t =$	·00010	8th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{15}}v = 60{\cdot}8$ $\frac{\sqrt{7}-\sqrt{6}}{\sqrt{15}}t =$	·00011	9th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}}v = 66{\cdot}1$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}}t =$	·00012	10th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{15}}v = 73{\cdot}2$ $\frac{\sqrt{5}-\sqrt{4}}{\sqrt{15}}t =$	·00013	11th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{15}}v = 83{\cdot}0$ $\frac{\sqrt{4}-\sqrt{3}}{\sqrt{15}}t =$	·00014	12th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{15}}v = 98{\cdot}5$ $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{15}}t =$	·00017	13th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{15}}v = 128{\cdot}3$ $\frac{\sqrt{2}-\sqrt{1}}{\sqrt{15}}t =$	·00022	14th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{15}}v = 309{\cdot}8$ $\frac{\sqrt{1}-\sqrt{0}}{\sqrt{15}}t =$	·00054	15th
Sum 1200·0	Sum $\frac{1}{480}$ or ·00208	sec.

Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM XVIII.

158. *To determine the Circumstances of Space, Penetration, Velocity, and Time, arising from a Ball moving with a given Velocity, and striking a moveable Block of Wood, or other Substance.*



Let the ball move in the direction AE, passing through the centre of gravity of the block B, impinging on the point C; and when the block has moved through the space CD, in consequence of the blow, let the ball have penetrated to the depth DE.

Let B = the mass or matter in the block,
b = the same in the ball,
s = CD the space moved by the block,
x = DE the penetration of the ball, and theref.
$s + x$ = CE the space described by the ball,
a = the first velocity of the ball,
v = the velocity of the ball at E,
u = veloc. of the block at the same instant,
t = the time of penetration, or of the motion,
r = the resisting force of the wood.

Then shall $\frac{r}{B}$ be the accelerating force of the block,
and $\frac{r}{b}$ the retarding force of the ball.

Now, because the momentum $B\dot{u}$, communicated to the block in the time $\dot{t}$, is that which is lost by the ball, namely, $-b\dot{v}$, therefore $B\dot{u} = -b\dot{v}$, and $Bu = -bv$. But when $v = a$, $u = 0$; therefore, by correcting, $Bu = b(a - v)$; or the momentum of the block is every where equal to the momentum lost by the ball. And when the ball has penetrated to the utmost depth, or when $u = v$, this becomes $Bu = b(a - u)$, or $ab = (B+b)\,u$; that is, the momentum

before the stroke, is equal to the momentum after it. And the velocity communicated will be the same, whatever be the resisting force of the block, the weight being the same.

Again, by forces, it is $u^2 = \frac{4grs}{B}$; and $-v^2 = \frac{4gr}{b} \times (s + x)$, or rather, by correction, $a^2 - v^2 = \frac{4gr}{b}(s + x)$. Hence the penetration or $x = \frac{b(a^2 - v^2) - 4grs}{4gr}$. And when $v = u$, by substituting u for v, and Bu^2 for $4grs$, the greatest penetration becomes $\frac{ba^2 - (B + b)u^2}{4gr}$; and this again, by writing ab for its value $(B + b)u$, gives the greatest penetration $x = \frac{Bba^2}{4gr(B + b)} = \frac{ba^2}{4gr} \times (1 - \frac{b}{B + b})$. Which is barely equal to $\frac{ba^2}{4gr}$ when the block is fixed, or infinitely great; and is always very nearly equal to the same $\frac{ba^2}{4gr}$ when B is very great in respect of b. Hence then

$$s + x = \frac{a^2 - u^2}{4gr}b = \frac{a^2 - \frac{a^2b^2}{(B + b)^2}}{4gr}b = \frac{B^2 + 2Bb}{(B + b)^2} \times \frac{a^2b}{4gr}.$$

And therefore $B + b : B + 2b :: x : s + x$,

or $B + b : \quad b \quad :: x : s$,

and $s = \frac{bx}{B + b} = \frac{Bb^2a^2}{4gr(B + b)^2}$.

159. *Exam.* When the ball is iron, and weighs 1 pound, it penetrates elm about 15 inches when it moves with a velocity of 1200 feet per second, in which case,

$\frac{r}{b} = \frac{a^2}{4gx} = \frac{1200^2}{4 \times 16 \times \frac{15}{12}} = \frac{300^2}{5} = 18000$ nearly.

When $B = 400$lb, and $b = 1$; then $u = \frac{ab}{B + b} = \frac{1200}{401} = 3$ feet nearly per second, the velocity of the block.

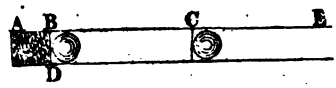
Also $s = \frac{Bu^2}{4gr} = \frac{400 \times 9}{4 \times 16 \times 18000} = \frac{1}{320}$ part of a foot, or $\frac{3}{80}$ of an inch, which is the space moved by the block when the ball has completed its penetration.

And $t = \frac{2s}{u} = \frac{2}{320 \times 3} = \frac{1}{480}$ part of a second; or $t = \frac{2s + 2x}{v} = \frac{\frac{2}{320} + \frac{30}{12}}{1200} = \frac{1 + 400}{1200.1600} = \frac{1}{480}$ part of a second nearly, the time of penetration.

For the circumstances relating to the motion of a block, suspended by, and vibrating on, an axis, when struck by a ball, see Tract 34, arts. 21 . . . . 31.

PROBLEM XIX.

160. *To determine the Circumstances relating to a given Ball discharged from a Cannon, with a given Charge of Powder.*



Let the annexed figure represent the bore of the gun; ABD being the part occupied with the gunpowder. And put

a = AB, the length at first occupied by the charge;
b = AE, the whole length of the gun bore;
c = ·7854, the area of a circle whose diameter is 1;
d = BD, the diameter of the ball, or of the bore;
e = the specific gravity of the ball, or weight of 1 cubic foot;
g = $16\frac{1}{12}$ or 16 feet, descended by a body in 1 second;
m = 230 oz, the pressure of the atmosphere on 1 sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
w = the weight of the ball; and let
x = AC, any variable distance of the ball from A, in moving along the gun barrel. Then,

First, cd^2 is = the area of the circle BD of the ball; therefore mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the force of the powder on BD.

But the force of the inflamed powder is supposed to be proportional to its density, and the density is inversely as the space it occupies; therefore the force of the inflamed powder on the ball at B, is to the force on the same at C, as AC is to AB; that is,

$x : a :: mncd^2 : \frac{mnacd^2}{x}$ = F, the motive force at C;

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, by forces, $v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x}$;
the fluent of which is $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. of x.

But, when $v = o$, then $x = a$; therefore, by correction, $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent; conseq. $v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the veloc. of the ball at C; or $v = \sqrt{(\frac{4gmnhcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ is the velocity with which the ball issues from the muzzle at B; where h denotes the length of the cylinder filled with powder; a being the length to the hinder part of the ball; which will be more than h when the ball does not touch the powder.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is $w = \frac{2}{3}cd^3e$; therefore $\sqrt{\frac{4gmnhcd^2}{w}} = \sqrt{\frac{6gmnh}{de}} = \sqrt{\frac{96.230.144nh}{de}} = 1783\sqrt{\frac{nh}{de}}$. Consequently the rule is

$v = 1783\sqrt{(\frac{nh}{de} \times \text{hyp. log. } \frac{b}{a})} = 2706\sqrt{(\frac{nh}{de} \times \text{com. log. } \frac{b}{a})}$.

When the ball is of cast iron, $e = 7400$, and the rule is $v = 31{\cdot}45\sqrt{(\frac{nh}{d} \times \text{log. } \frac{b}{a})}$ for the velocity of the iron ball.

Or, when the ball is of lead, then $e = 11325$, and $v = 25{\cdot}42\sqrt{(\frac{nh}{d} \times \text{log. } \frac{b}{a})}$ for the velocity of the leaden ball; in which two theorems, a, b, d, h, may be taken in any measures, either feet or inches, &c.

161. *Exam.* For an example, it has been found that the medium velocity of the gun n° 2, with a charge of 4 ounces of powder, has been about 1180 feet. Now, the ball being 1·96 inches in diameter, and the charge of powder with its containing bag occupying 3·45 inches of the cylinder, but the powder alone only 2·54; the numeral values of the letters in the theorem $v = 31{\cdot}45\sqrt{(\frac{nh}{d} \times \text{log. } \frac{b}{a})}$, will be $a = 3{\cdot}45$, $b = 38{\cdot}43$, $d = 1{\cdot}96$, $h = 2{\cdot}54$; and if we suppose $n = 1000$, as assumed by Mr. Robins; these substituted in that theorem, give $v = 1159$, being about 21 feet too little.

162. Such then is the solution of the problem in its most

simple state, v the velocity, and n the first force of the inflamed powder, being the unknown quantities to be determined. But, several circumstances are omitted in this solution, which would diminish the velocity of the ball; and though, when taken separately, they may amount to but little, yet, when taken all together, they may be too considerable to be entirely neglected. The first of these, that may be noticed, is the counter pressure of the atmosphere or external air, which ought to be taken into the account; for, so long as the ball continues in the cavity AE, it is pressed back, and its motion retarded, by the pressure of the incumbent air on the forepart of the ball. This pressure is equal to that of a column of mercury 30 inches high, or equal to mcd^2 as found above; which retarding force is to the weight of the ball w, as $\frac{mcd^2}{w}$ to 1. Therefore, instead of the first fluxional equation, it will now be $v\dot{v} = \frac{2gmcd^2}{w} \times (\frac{an\dot{x}}{x} - \dot{x})$, the correct fluent of which gives $v^2 = \frac{4gmcd^2}{w} \times (an \times \text{hyp. log.} \frac{x}{a} + a - x)$; and when $x = \text{AE} = b$, then $v = \sqrt{(\frac{4gmcd^2}{w} \times an \times \text{hyp. log.} \frac{b}{a} + a - b)}$ is the velocity with which the ball issues at the muzzle of the gun.

But $\sqrt{\frac{4gmcd^2}{w}} = \sqrt{\frac{6gm}{de}} = \sqrt{\frac{96m}{de}} = \sqrt{\frac{96 \cdot 230 \cdot 144}{de}} = \frac{1783}{\sqrt{de}}$; therefore the theorem is $v = \frac{1783}{\sqrt{de}} \times (an \times \text{hyp. log.} \frac{b}{a} + a - b)$.

163. By taking the same example as before, and computing it by this rule, the result comes out only the 366th part less than before, or about 8 feet less, being too inconsiderable to be regarded. Should we also take into the account the effect of the air before the ball, which is condensed there by the resistance from the pressure of the atmosphere, it will probably amount to about an equal quantity, or little more. So that the two together may be estimated at about the 180th part of the whole.

164. There are also several other causes of the diminution of the velocity, besides the two above-mentioned. One of these is the friction of the ball in moving along the cylin-

drical bore of the piece. This indeed cannot amount to much: but, whatever it is, there does not appear to be any exact method of computing it. Another cause of diminution, is the loss of force by the elastic fluid escaping by the vent and windage of the gun; and this is likely to be very considerable, as the fluid will issue out by the vent, and by the sides of the ball, with its whole velocity, which is at a rate vastly more than that of the ball itself. Some estimate of the loss of velocity by this cause, might be deduced from some of the earliest part of our experiments; but a more accurate calculation of it must be deferred, till we come to determine with what velocity it is that the elastic fluid expands itself.

165. Other causes of diminution in the velocity are, the gradual firing of the powder, also the unequal motion and density of the elastic fluid behind the ball, and lastly the weight of the powder itself, which must require part of the force to give it so violent a motion. The investigation of the theorem proceeds on the supposition that all the powder is fired the first instant. But, though the time of firing is manifestly very small, yet it must be acknowledged that it is not all inflamed in an instant, both from the circumstance, in the higher charges, of many grains of powder being blown out unburnt, and the velocity of the ball being very sensibly below what it ought to be, on the supposition of the total and instantaneous inflammation. A considerable diminution must therefore result on this account, without admitting of a theoretical investigation, and without the means of an estimation, except from the defect of its due force in the experiments.

166. Again, the motion of the inflamed fluid is slowest next the breech, and gradually quicker all the way, from thence to the ball, where it is the quickest. For the same reason also, the fluid is densest and strongest next the breech, and all the way rarer and weaker to the ball, where it is the weakest. Besides, the fluid is supposed to act upon the ball,

and to urge it, with its whole force and velocity; whereas in fact it only acts with its relative velocity, or by the excess above the velocity of the ball. On all these accounts then, it is manifest that the velocity in the experiment must be greatly diminished, without receiving a previous theoretical calculation, to estimate its effects.

167. Lastly, in the calculation it is supposed that the whole strength of the powder is exerted in urging the ball forward; whereas some part of the force must be employed in giving motion to the gross particles of the powder, and even the generated fluid itself; which consequently ought to be deducted from that which was supposed to act on the ball, or its weight, either in whole or part, added to the ball, by which means the motion of this will be still further diminished. On all these, and perhaps other accounts also, it is manifest that the experimented velocities of balls must be far below what they ought to be, if the whole force of the inflamed powder was employed and spent only upon the ball in giving it motion.

168. To renew the calculation of the velocity then, so as to include in it, as well as that of the ball, both the weight of the charge of powder, and the flannel bags in which it was contained. Now, because the elastic fluid of the inflamed powder occupies, at all times, the part of the gun bore behind the ball, the middle of it, or nearly its centre of gravity, will be moved with only half the velocity that the ball moves with; and this will require the same force as half the weight of the powder, moved with the whole velocity of the ball. Therefore, in the result before derived, we are now, instead of the weight of the ball w, to substitute the quantity $p + w$, and then, instead of the first theorem, it will now become $v = \sqrt{(\frac{4gmnhcd^2}{p+w} \times \text{hyp. log.} \frac{b}{a})}$ for the velocity of the ball, p denoting half the weight of the powder and bag.

But when the given numbers are substituted for the letters g, m, c, and the hyperbolic logarithms are reduced to the

common ones, then the same formula becomes
$v = \sqrt{(\frac{2210nhd^2}{p+w} \times \text{com. log.} \frac{b}{a})}$, or it is also nearly
$v = 47\sqrt{(\frac{nhd^2}{p+w} \times \log. \frac{b}{a})}$, the same velocity; or it is
$v = 46{\cdot}1\sqrt{(\frac{nhd^2}{p+w} \times \log. \frac{b}{a})}$, including the allowance for the counter-pressure of the air.

169. But now, as we have determined by numerous experiments, detailed in the 34th Tract, the actual velocities of balls in a great multitude of instances, by substituting then any of those velocities for v, in the last theorem, it is manifest that the equation will give us the value of the unknown quantity n, which has hitherto been assumed to denote the first force of fired gunpowder; the values of all the other letters in the formula being known. Considering all the other quantities then as given, and reducing the last equation to find n, it gives us $n = \frac{p+w}{2180hd^2}v^2 \div \log.$ of $\frac{b}{a}$ for the general value of the first force of the elastic fluid, or how often it is stronger than the pressure of the atmosphere; a quantity which has been assumed equal to 1000 by Mr. Robins, but imagined by several other persons to be some larger number. In the theorem, d and h denote inches, v feet, and p and w ounces.

170. Now, to make application of this theorem to some of the cases in Tract 34, to the end of the year 1786, for the three cases of 4, 8, and 16 ounces of powder, with all the four different lengths of guns. We have seen that the ball there employed, was of 1·96 inches in diameter, and its medium weight 16 oz 13 dr = 16·81 oz. It has been found also, that the weights of the bags containing the 4, 8, and 16 ounces of powder, were 8 dr, 12 dr, and 1 oz 5 dr; these then being added to the respective weights of powder, the sums 4·5 oz, 8·75 oz, 17·31 oz, are the correspondent values of $2p$; and their halves, 2·25 oz, 4·38 oz, and 8·66 oz, are the values of the quantity p for those three charges; these then being added to 16·81, the constant weight of the ball, there result the three values of $p + w$ for the said three

charges of powder, which values therefore are 19·06, and 21·19, and 25·47 ounces. It may be remarked also that the lengths of the gun bores, when corrected for the spherical hollow in the bottom of the bore, is, for the four guns, 28·53, 38·43, 57·70, and 80·23. Also, that the length of the three charges, when corrected in like manner, are 3·45, 5·99, and 11·07, for the powder and bag together; but for the powder alone only 2·54, 5·08, and 10·16. So that a general synopsis of all these dimensions and data, with the correspondent velocities of the ball, will be as in the following table.

The Gun.		Charge of Powder.			Wt. of ball and charge, or values of $p+w$.	Velocity, or the values of v.	First force, or the value of n.
N°	Length or value of b.	Wt. in ounces.	Length or value of a.	Length or value of h.			
1	28·53	4	3·45	2·54	19·06	1100	1182
		8	5·99	5·08	21·19	1340	1319
		16	11·07	10·16	25·47	1430	1531
2	38·43	4	3·45	2·54	19·06	1180	1192
		8	5·99	5·08	21·19	1480	1440
		16	11·07	10·16	25·47	1660	1526
3	57·70	4	3·45	2·54	19·06	1800	1238
		8	5·99	5·08	21·19	1790	1622
		16	11·07	10·16	25·47	2000	1670
4	80·23	4	3·45	2·54	19·06	1870	1231
		8	5·99	5·08	21·19	1940	1664
		16	11·07	10·16	25·47	2200	1684

Then, by computing the values of the force n, by the preceding formula, from the several data contained in the first seven columns of this table, those values come out as they are here arranged in the 8th or last column.

171. Here it is observable that the values of n, for the first force of the elastic fluid, are considerably various, both for all the guns and for all the charges of powder; several of which circumstances can be easily accounted for, and some of them appreciated. And first, it must be considered

that these numbers, for the value of n, ought to exhibit the first force of the inflamed powder, when it is supposed to occupy the space only in which the bare powder itself lies; whereas it is manifest, that the condensed fluid of the charge, in these experiments, occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag together: which exceeds the former space, or that of the powder alone (besides the different lengths in the bore, which has been accounted for by using h for a in the theorem) at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest then that the force was diminished in the same ratio on that account. Now, by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity by which the diameter of the circle of the powder within the bag, was less than that of the gun bore. But the diameter of the gun bores was 2·02 inches; therefore, deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag: and because the areas of circles are to each other as the squares of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408, or as 97 to 102; therefore, on this account alone, the numbers before found, for the values of n, in the last column of the table, must be increased in the ratio of 97 to 102, or nearly the $\frac{1}{19}$ part; which increase being made, the numbers for the value of n will be as in the annexed tablet, so far as includes the effect of this cause of increase.

Powder.	The Guns.			
oz	1	2	3	4
4	1244	1255	1303	1296
8	1388	1521	1707	1752
16	1612	1606	1758	1773

172. There is another circumstance, which occasions the space, at first occupied by the inflamed powder, to be larger than that at which it has been taken, in the foregoing calculations; and that is the difference between the content of a sphere and cylinder. For, the space supposed to be occupied at first by the elastic fluid, was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little, by the difference between the content of half the ball, and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ of the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is nearly 0·33. Hence then it appears that the length of the cylinders, at first supposed to be filled by the dense fluid, viz, 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; and hence it follows that, on this account also, all the numbers before found for the value of the first force n, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40. When this is done, the numbers for the value of n, so increased, will be as in the annexed tablet.

Powder.	The Guns.			
oz	1	2	3	4
4	1359	1376	1427	1421
8	1457	1612	1793	1840
16	1660	1654	1811	1826

173. Another source of diminution of the force, is the waste of the fluid by the vent and the windage. These diminutions, from the extreme rapidity of the fluid, must be very considerable, though the effects are very difficult to be computed with accuracy. This loss of force arises from hence, that while the ball passes along the gun cylinder, the elastic fluid constantly escapes, not only at the vent or touchhole, but also between the ball and the sides of the cylinder; so that the elastic force must always be greater than those

above found by computation from the experimented velocities. This fluid, on account of its great elasticity, moves much quicker through the above-mentioned openings, than the ball ever can move. Hence the diminution of the impelling force must be considerable, and must necessarily make the velocity of the ball much less than what it would otherwise be. In order to assign the quantity of the decrease from this cause, it is necessary to know with what velocity the compressed elastic air, in the gun, passes through the vent, and by the sides of the ball. This degree of velocity can never be communicated to the ball, since the powder acts on the ball only by its relative velocity, that is, by so much as the velocity of the ball is less than that of the flame would be, if there was no ball at all, or if it was suddenly annihilated. Since the ball, by its own mass, resists the motion of the flame, as well as meets with resistance from external causes, it is easy to conceive that the ball can never acquire so great a velocity as the flame would acquire, if no obstruction lay in its way. Hence it is obvious, that the greatest velocity which the powder can communicate to the ball, will always be less than that which the pure flame would acquire, in proportion to the weight of the ball it impels, and the resistance the ball meets with. And hence, it is of the greatest importance to obtain a knowledge of that velocity. That it is immensely great, is evident from the circumstance of its communicating such high velocities to the heaviest balls and shells, being probably not less than 3 or 4 times that of the balls in the more ordinary cases. There may be several ways of determining this, either by theory or by experiments. But, preparatory to that determination, we shall first enquire what aid towards it may be further derived from our own experiments, especially by another correction of our theorem for the ball's velocity, on account of the waste of the elastic fluid by the vent and windage.

174. This subject has before been adverted to in arts. 77 and 113 of the 34th Tract. It there appears that, upon an average, about the $\frac{1}{4}$ part of the force is lost by the difference

of $\frac{1}{10}$ of an inch in the windage, viz, between the balls of 1·96 and 1 86 diameter, that of the cylinder of the gun being 2·02. And the quantity lost being as the spaces, and the spaces as the differences of the squares of the diameters; therefore, as $1{\cdot}96^2 - 1{\cdot}86^2 : \frac{1}{3} :: 2{\cdot}02^2 - 1{\cdot}96^2 : \frac{16}{77}$ nearly = ·21 nearly; and this is the part of the force lost by the windage between 2·02 and 1·96 diameters. There remains yet to be computed the waste by the vent. If we estimate this in proportion to its magnitude, which will not be far from the truth; and its diameter being nearly $\frac{1}{5}$ of an inch = ·2; therefore as $1{\cdot}96^2 - 1{\cdot}86^2 : \frac{1}{3} :: {\cdot}2^2 : {\cdot}04$ nearly, the waste of force by the vent only. The two together make ·25 or $\frac{1}{4}$, for the waste by the windage and vent with the usual ball of 1·96 diameter. On this account then we must increase each of the numbers in the last tablet by the $\frac{1}{4}$ part, by which means they will then become as here annexed, for the several values of the first force n.

Powder.	The Guns.			
oz	1	2	3	4
4	1700	1720	1784	1776
8	1821	2015	2241	2282
16	2075	2068	2264	2300

175. Such then are finally the numbers denoting the value of n, the first force of the inflamed elastic fluid, including the effect of heat, when occupying the space only filled by the powder alone, before it was fired, when most of the material allowances are made; and which we thus find is about double of what Mr. Robins supposed it to be. With each charge of powder, there is but very little difference among all the four guns, being of equal diameter, though of very different lengths; the numbers increasing but a very little with the longer guns, probably arising from the circumstance of the powder acting rather a longer time on the ball in the long guns, than in the shorter; a circumstance which very well accounts for the small difference in the effect. But a

greater difference takes place among the effects of the different charges, with each of the guns, as in each case there is a pretty uniform increase with the charge of powder; a circumstance which probably happens from the greater degree of heat likely to take place with the greater quantity of powder, and which may therefore very well account for the differences here adverted to. So that all the numbers, now finally deduced, seem to possess all the favourable appearances that might be expected to happen from the causes most likely to produce such effects.

176. Let us now advert to other consequences resulting from the foregoing conclusions. We have found the strength of inflamed gunpowder, when fired, or while the flame occupies only the same space as the powder did before it was fired, on a medium to be about 2000 times the elasticity of common air, or 2000 times the pressure of the atmosphere, when the barometer stands at 30 inches, that is, equal to the weight of a column of mercury, of equal base, and of the height of 60000 inches, or 5000 feet, being almost a mile in height; that is, including the effect of the heat of the flame. Now, it has been shown by Mr. Robins, and several other philosophers, that the elastic air generated by the firing of any parcel of gunpowder, when cooled to the temperature of common air, and made to fill a space of about 244 times greater than the bulk of the powder, was then reduced exactly to the same strength or elasticity as common air; and that therefore, when cooled, and filling only the first space of the unfired powder, the elasticity being at least in as high a proportion as the density, its elasticity must be then equal to 244 times that of natural air. And having found that a heat equal to that of red-hot iron, will increase the elasticity of air $4\frac{1}{11}$ times; therefore, supposing the heat of inflamed gunpowder to be equal to that of the red-hot iron, he concluded that the elasticity of the heated elastic fluid would be equal to $244 \times 4\frac{1}{11}$ or 998 times, or in round numbers about 1000 times, greater than that of the atmosphere.

177. On this principle it was that Mr. Robins made all his experiments, and performed all his calculations in gunnery. But it is manifest that this method of guessing at the degree of heat of the flame must be very uncertain and unsatisfactory, as much below the truth, since all our notions and experience of the heat of inflamed powder convince us, that it is higher than that of red-hot iron; and indeed it has clearly appeared from our experiments, that its heat is at least double of that of red-hot iron, and that it increases the elasticity of the elastic fluid more than 8 times. Mr. Robins, as before mentioned, stated that he computed the elastic fluid produced by the fired powder, when cooled, to be 244 times denser and stronger than common air: this however is a radical error, the effect of which runs through all his after calculations; and it happened in increasing the number 520, by its $\frac{1}{15}$ part, erroneously stating it thereby to become 575, instead of only 555; in consequence it happened that he brought out the number 244, instead of 236, which is the true measure of the elasticity of the cool elastic fluid, when occupying the same space as its producing powder. But it has been found, from our experiments, that by the heat of the flame, the elasticity is increased to 2000 times that of the atmosphere, which is an increase of almost 9 times the force, by means of the heat of the inflamed powder, instead of 4 times only.

178. We may hence also deduce the amazing degree of condensation of the elastic air in the nitre and gunpowder, and the astonishing force experienced by its explosion. It has been found by Mr. Robins and other philosophers, that $\frac{3}{10}$ of the mass of the powder consists of the pure condensed air, or that the weight of the condensed air is equal to $\frac{3}{10}$ of the whole composition. But the whole composition of the powder consists of 8 parts by weight, of which 6 parts are nitre, 1 sulphur, and 1 charcoal; of which the nitre, or $\frac{3}{4}$ of the composition, furnishes the whole of the condensed air, while the sulphur and charcoal only give the fire that pro-

duces the explosion. But $\frac{3}{10}$ of the whole mass of 8 parts, is equal to $\frac{4}{10}$ of the 6 parts or nitre; that is, $\frac{4}{10}$ or $\frac{2}{5}$ of the nitre consists of condensed air, or the weight of the air is to the weight of the gross matter in the nitre, as 4 to 6, or as 2 to 3; and these two parts it is probable are of equal density or specific gravity. But the specific gravity of nitre is 1900, that of water being 1000, and of air 1·2, which is contained in 1900, as much as 1583 times; that is, the air in the nitre must be condensed the amazing quantity of 1583 times, if its specific gravity be equal to the compound nitre itself!

179. The condensation of the air, in the solid nitre, may be otherwise determined thus. The weights of the three solid ingredients, nitre, sulphur, and charcoal, being as the numbers 6, 1, 1, and their specific gravities respectively 1900, 1810, and 400 nearly, water being 1000, or they are as 19 and 18 and 4 nearly, when water is 10. Therefore the magnitudes of these ingredients are as $\frac{6}{19}$, $\frac{1}{18}$, $\frac{1}{4}$, or as the numbers 216, 38, 171, when brought to a common denominator 684. The first fraction $\frac{216}{684}$ is nearly equal to the sum of the other two $\frac{38}{684}$ and $\frac{171}{684}$; that is, the magnitude of the nitre in the compound, is $\frac{1}{2}$ of the whole magnitude. But the condensed air is $\frac{2}{5}$ of the nitre; and $\frac{2}{5}$ of $\frac{1}{2} = \frac{1}{5}$; therefore the magnitude of the air is $\frac{1}{5}$ of the whole, considering them all as in a solid mass. But now the specific gravity of this mass will be found by dividing the weight 8, by the magnitude $\frac{216}{684} + \frac{38}{684} + \frac{171}{684} = \frac{425}{684}$, that is $8 \div \frac{425}{684} = 12{\cdot}8$ is the specific gravity of the solid mass, while that of the same when grained as powder for use, is only 9·37; then as 937 : 128 : : 236 (the density of the elastic fluid when occupying the space of the grained powder) : 322 the density of the same in the space of the solid. But the space occupied by the fluid is $\frac{1}{5}$ of the whole: therefore $322 \times 5 = 1610$ is the condensation of the same, or the number of times it is denser than common atmospheric air, and is nearly the same as before found by the other method. That is, the air is condensed in the nitre, about 1600 times, nearly double the density of

water; which may well be considered as probably the greatest degree of compression that air is capable of. Hence it may be perceived that a prodigious force must be exerted by nature in generating nitre: and as this great force actually exists in nature, it is very probable that the air in the nitre is thus compressed into the most dense state possible; and in this consists the similitude among the different particles of nitre.

180. It remains now then to determine the velocity with which the elastic flame expands itself and moves. If the whole substance of the powder was changed into an elastic fluid at the instant of the explosion, then, from the known elasticity of this fluid deduced from our experiments, in the foregoing articles, and its known density, we might determine the velocity with which it would begin to expand, and could thence discover its future augmentations in its progress through the gun cylinder. But, as it is probable even that the powder does not inflame all at once; and as it is certain that the greatest part of it, viz. $\frac{7}{10}$, consists of gross matter not convertible into an elastic fluid; which matter will in the explosion be mixed with the elastic part, and will by its weight retard the activity of the explosion, and yet they will not be so uniformly mixed, and intimately united, as to be moved with one common motion; but the gross part will be less accelerated than the elastic, and some of it will not even be carried out of the barrel, as appears by the quantity of unctuous matter which adheres to the inside of all fire-arms, after they have been discharged. And besides, it appears not that even the elastic fluid itself is uniformly diffused through all the space of the gun cylinder; but rather, on the contrary, that the elastic fluid must be more compressed and dense in the after parts, and less in front. From all these circumstances it is manifest that some uncertainty must attend even the application of any theorem to our best experiments, such as the preceding formula $v = 46{\cdot}7\sqrt{(\frac{nhd^2}{p+w} \times \log.\frac{b}{a})}$ for the value of the ultimate velocity v, or the

form $n = \frac{p + w}{2180hd^2}v^2 \div \log. \frac{b}{a}$ for the value of the initial force n. For, though these two thus depend the one upon the other; and the one of these, as v, be very well determined by the experiments; yet the formula includes a quantity p, the exact value of which it is impossible to assign, but must remain very uncertain, being only guessed at by judging from circumstances as near as may be. In these formulæ, p has been assumed equal to half the weight of the powder and bag; a portion which, from the uncertain circumstances before mentioned, must evidently be too large a quantity, because it presupposes a uniform diffusion and density throughout the mass of inflamed powder, both of the gross and the elastic parts. Such as it is however, we shall proceed to employ the first of these formulæ to determine the velocity of the flame, and afterwards make such corrections in the results as other reasons may happen to suggest.

181. Now, in the formula $v = 46{\cdot}7\sqrt{(\frac{nhd^2}{p + w} \times \log. \frac{b}{a})} = 46{\cdot}7d\sqrt{(\frac{nh}{p + w} \times \log. \frac{b}{a})}$ giving the ultimate velocity of the ball, whose weight is w, it is manifest that the less the weight w, in the denominator, is, the greater will be the value of the velocity v, of the ball, and the fore part of the elastic fluid, which pursues and urges it on; therefore, by supposing the value of w diminished till it quite vanishes, or becomes equal to nothing, the formula must then give the velocity of the expanding elastic fluid itself, subject only to the uncertainty of the assumed part p. Now, by omitting the weight w in that theorem, it becomes $v = 46{\cdot}7d\sqrt{(\frac{nh}{p} \times \log. \frac{b}{a})}$; or, substituting 1·96, the value of d, instead of it, the formula becomes $91{\cdot}532\sqrt{(\frac{nh}{p} \times \log. \frac{b}{a})}$; by which we shall now proceed to compute the values of v, according to the different values of all the quantities a, b, n, h, p, as before assigned, which several values, for the four guns, are as in the subjoined table.

The Gun.		Charge of Powder.					Velocity of flame,
N°	length b	wt. oz.	Values of a	h	p	n	v
1	28·53	4	3·45	2·54	2·25	1700	3841
		8	5·99	5·08	4·38	1821	3464
		16	11·07	10·16	8·66	2075	2896
2	38·43	4	3·45	2·54	2·25	1720	4127
		8	5·99	5·08	4·38	2015	3976
		16	11·07	10·16	8·66	2068	3305
3	57·70	4	3·45	2·54	2·25	1784	4544
		8	5·99	5·08	4·38	2241	4629
		16	11·07	10·16	8·66	2264	3995
4	80·23	4	3·45	2·54	2·25	1776	4791
		8	5·99	5·08	4·38	2282	4999
		16	11·07	10·16	8·66	2300	4405

Then, by computing the velocity for every one of the twelve cases, by the last formula, they come out as they are arranged in the last column of this table. And here it is seen that these numbers, for the velocity of the elastic flame, gradually increase from the shortest gun, to the longest, from the number 3000 to 5000, as might be expected to happen from the different lengths of the guns, but in a far smaller degree, like the numbers for the value of n, in the last column but one. The four mediums, among the three numbers for each gun, are 3400, and 3803, and 4389, and 4732; also the medium among all these is 4081. But, by an experiment of a different kind, Mr. Robins found the velocity of the flame come out as high as 7000, which is probably nearer the truth, as we suspected, before making the computation, for the reasons before mentioned, and chiefly by taking so much as half the weight of the charge for the value of p, while it is more likely that it should not be more than $\frac{1}{4}$ or $\frac{1}{3}$ so much as has been assumed for p. If a calculation be made with $\frac{1}{3}$ of the same value of p, the result comes out between 7 and 8 thousand, which may probably be nearer the true quantity;

and with $\frac{2}{3}$ of p, or $\frac{1}{3}$ the whole weight of the charge, the velocity is between 6 and 7 thousand, which is probably the nearest of any.

182. Another consequence to be derived from our theorem for the velocity of the ball, is the proper charge for any given gun, so as to communicate to the ball the greatest velocity possible. It is not difficult to perceive that the velocity will not be increased by increasing the charge beyond a certain degree; because, when the barrel is almost full of powder, the ball will be out of the piece before the charge has time to give it the full velocity; and, on the other hand, when the charge is very small, it is too weak to give the ball a sufficient impulse. So that, by increasing the charge gradually from the smallest, the velocity communicated to the ball will be gradually increased, till it arrive at a certain degree; after which, the velocity will be gradually decreased again, as the charge is more increased, till the barrel is quite filled with the powder. Hence it follows that, in every gun, there is a certain charge which will give the greatest velocity to the ball; and that by either increasing or diminishing the charge, the motion of the ball will be diminished. Now it is evident that the knowledge of this charge is of great importance in artillery: for, by this means the artillerist is enabled to discharge the ball with the greatest velocity; and sometimes to save much powder, by knowing that a greater charge would not communicate so great a motion, but perhaps much less. Therefore, to find the charge which will give the greatest motion to the ball, we must make the formula a maximum which expresses the velocity, or its fluxion equal to nothing, considering the length a of the charge as the varying quantity.

183. If, for this purpose, we take here the first formula, $v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log.} \frac{b}{a})}$; then by squaring and cancelling the constant factors, we obtain $a \times$ hyp. log. $\frac{b}{a}$ a maximum, or the hyp. log of $(\frac{b}{a})^a$ a maximum. The fluxion

of this is $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$; hence the hyp. log. of $\frac{b}{a} = 1$, and $\frac{b}{a} =$ the number whose hyp. log. is 1, that is $\frac{b}{a} = 2{\cdot}71828$, and $\frac{a}{b} = \frac{1}{2{\cdot}71828}$, or the length of the charge a is rather more than $\frac{1}{3}$ of the length of the gun bore. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer as 7 to 19; that is, the length of the charge, to communicate the greatest velocity, by the above formula, is the $\frac{4}{11}$ of the length of the bore, or nearer $\frac{7}{19}$ of it, or about $\frac{1}{3}$. By our experiments it has been found (art. 124 Tract 34) that the charge for the greatest velocity, is but little less than that which is here computed from theory, the correspondent parts there found, for the four guns, being nearly $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying as the gun is longer, which allows more time for the greater quantity of powder to be fired, and for the flame to act on the ball, before it is out of the bore.

184. But we shall be likely to come nearer to the experimented quantity, if, instead of the first or incorrect formula, we employ the more exact one, which includes the weight of a portion of the powder, as well as that of the ball; viz, the theorem $v^2 = \frac{4gmnacd^2}{p + w} \times$ hyp. log. $\frac{b}{a}$, or, omitting the constant factors, $\frac{acd^2}{p + w} \times$ hyp. log. $\frac{b}{a}$ must be a maximum. Now it will be convenient here to express p and w in terms of a and d. And first, the content of the ball being $\frac{2}{3}cd^3$ in cubic inches; and the cubic inch of cast iron weighing 4·3 ounces; therefore $4{\cdot}3 \times \frac{2}{3}cd^3 = 2{\cdot}8\frac{2}{3}cd^3$ is $= w$ the weight of the ball in ounces nearly. Also the length of the cylinder of the powder and bag being a, its content will be acd^2 inches; and the weight of the cubic inch of the same being half an ounce nearly, or rather ·54, therefore $\cdot 54acd^2$ is $= 2p$ the weight nearly, or $p = \cdot 27acd^2$. Substituting now these values of p and w in the denominator of the maximum above mentioned, it becomes $\frac{acd^2}{\cdot 27acd^2 + 2{\cdot}8\frac{2}{3}cd^3} \times$ hyp. log. $\frac{b}{a}$, or $\frac{a}{10{\cdot}6d + a} \times$ hyp. log. $\frac{b}{a}$ the maximum. Then putting the fluxion of this

quantity $= 0$, the equation reduced gives hyp. log. of $\frac{b}{a} = \frac{10{\cdot}6d + a}{10{\cdot}6d} = 1 + \frac{a}{10{\cdot}6d}$, a general expression for the value of the hyp. log. of $\frac{b}{a}$, in assuming the whole value of p, or half the weight of the powder in the formula.

185. Now, though it might be very easy to find the value of a in this equation, for any particular values of b and d, by means of the rule of double position; yet it may be proper to obtain an equation, and a general value of the length of the charge a, out of logarithms. And for this purpose it may be convenient to employ Dr. Halley's approximation for a number from a given logarithm, which is this, the number is $\frac{b}{a} = \frac{2+l}{2-l}$, where $\frac{b}{a}$ is the required number, and l its hyperbolic logarithm. Now, in the present case the quantity l denotes the fraction $\frac{10{\cdot}6d+a}{10{\cdot}6d}$; which therefore being substituted for l, that number becomes $\frac{b}{a} = \frac{31{\cdot}8d+a}{10{\cdot}6d-a} = 3 + \frac{4a}{10{\cdot}6d-a}$. Hence it appears that, in all cases, the value of a is less than $\frac{1}{3}$ of b, and that so much the more as the charge is higher. The diameter d being $= 1{\cdot}96$, which substituted for it, the last equation becomes $\frac{b}{a} = \frac{63\frac{1}{7}+a}{21-a}$; which reduces to this quadratic equation, $a^2 + (63 + b)\,a = 21b$. This applied to our first gun, where $b = 28{\cdot}53$, it gives $a = 6\frac{1}{7}$; hence $\frac{b}{a} = 4{\cdot}6$ nearly; which by the experiment was $\frac{14}{4} = \frac{7}{2} = 3\frac{1}{2}$. —But when the same quadratic equation is applied to the 4th gun, in which $b = 80\frac{1}{4}$, it brings out $a = 11$ nearly; consequently $\frac{b}{a} = 7\frac{1}{4}$ nearly, but which by the experiment was $\frac{27}{4} = 6\frac{3}{4}$. So that, in every case, the theorem brings out the value of a rather too great, when the whole value of p, or the half weight of the charge, is used in the denominator $p + w$.

186. But, we before found, in determining the velocity of the inflamed elastic fluid, that the properest value of the quantity p, is not equal to half the weight of the charge, but rather to $\frac{1}{3}$ of it, or to $\frac{2}{3}p$. By using this quantity then, in-

stead of p, as above, and repeating the process, the final equation comes out nearly $a^2 + (96 + b)\,a = 32b$; where the numeral coefficients are to those in the former case, as 3 to 2, the same as the two values p to $\frac{2}{3}p$ here assumed: and so it will always be in the cases of other assumptions for that of p. Or the equation will be $a^2 + (48 + b)\,a = 16b$; when a and b are estimated in calibers of the gun. Then, by computing the values of a in this equation, for all the four different lengths of our guns, they come out very near the same as those before found in the experiments. Also, from the same equation the following table has been computed, for the lengths of the charges proper to give the greatest velocity by several different guns, the lengths being estimated in calibers or diameters of their bores, by the common difference of 5.

Length of the piece in calibers.	Length of the charge in calibers.
b	a
5	1·47
10	2·64
15	3·60
20	4·42
25	5·12
30	5·73
35	6·27
40	6·75
45	7·19
50	7·58

187. All these calculations, from theory, agree very nearly with the result of the experiments; a coincidence which is at once a proof and confirmation mutually of the one and the other. Mr. Euler has given a similar table from theory, in which the lengths of the charges are all very erroneous, being too great by quantities in a regular increasing series, from the ratio of 1 to 1·64 the least, to that of 1 to 1·90 the greatest, being almost double. And in many other cases the

calculations of the same author, though extremely ingenious, are equally erroneous, chiefly for want of a knowledge of the true resistance of the air, to bodies when moved with great velocities.

188. From what has been done in the last two articles, we can now deduce a much more convenient theorem for finding the velocity in all cases, as well as the greatest velocity communicated to the ball, from any given gun, and with any given charge of powder. In those articles, the quantities employed, besides the constant numbers, are or may be all counted in calibers of the gun; for the diameter d is 1 caliber, and a and b, the length of the charge and of the gun, can be as conveniently estimated in calibers, as in any other measures whatever. The general theorem being $v^2 = \frac{4gmn}{12} \times \frac{acd^2}{w+\frac{1}{3}w'} \times \text{hyp.log.}\frac{b}{a} = \frac{64 \times 230n}{12} \times \frac{acd^2}{w+\frac{1}{3}w'} \times \text{hyp.}$ $\log.\frac{b}{a} = 1226n \times \frac{acd^2}{w+\frac{1}{3}w'} \times \text{hyp. log.}\frac{b}{a}$, or $v^2 = 2824n \times \frac{acd^2}{w+\frac{1}{3}w'} \times \text{com. log.}\frac{b}{a}$, or $v^2 = 8473n \times \frac{acd^2}{3w+w'} \times \text{com.log.}\frac{b}{a}$; where w denotes the weight of the iron ball, w' the weight of the powder, n the first strength of the powder, or how many times it is stronger than the elasticity of common air, and $c = \cdot 7854$. But now, the diameter of the ball being d, its content will be $\frac{2}{3}cd^3$ cubic inches, and its weight $w = 4\cdot 3 \times \cdot\frac{2}{3}cd^3 = \frac{8}{3}^6cd^3$. And the diameter of the powder being d also, and its length a; therefore its content is acd^2, and its weight $\cdot 54225acd^2$. Then, substituting these values of w and w' in the above formula, it becomes by reduction $v^2 = 15625n \times \frac{a}{16d+a} \times$ log. of $\frac{b}{a}$ nearly. Or, putting $q = \frac{a}{d}$ the number of calibers or diameters contained in a the length of the charge, the theorem becomes $v^2 = 15625n \times \frac{q}{16+q} \times$ log. of $\frac{b}{a}$, or $v = 125\sqrt{(\frac{nq}{16+q} \times \text{log. of } \frac{b}{a})}$; being the simplest and most easy theorem for the ball's velocity that has yet been found, and is besides general for all diameters and lengths of guns whatever, all the dimensions being taken in calibers or diameters of the bore only. The quantity n might

also have a numeral value given to it; for, though the value of n be a little various, for the different charges and guns, being between the numbers 1700 and 2300 in the last determination of them in art. 174. But as that statement of those values of n, was the result of experiments made with the charges of powder put into bags, which renders the length of the charge always considerably more than the real length which the powder alone would occupy in the gun cylinder, especially in the lower charges; and as the theorem, which is just above given, supposes a to denote the length of the powder alone, without any including bag; it may therefore be best to take a medium among the values of n for the highest charge, as shown in the last line of the table in the determination above referred to, which numbers, for the four guns, run from 2075 to 2300, the medium among the four being nearly 2200, the square root of which is 47 very nearly. This then being substituted for $\sqrt{n}$ in the last theorem, it becomes finally $v = 5875\sqrt{(\frac{q}{16+q} \times \text{log. of } \frac{b}{a})}$, for the velocity of the ball in all cases, both accurate, and in its most simple form. Or it may be $v = 5875\sqrt{(\frac{a}{16+a} \times \text{log. of } \frac{b}{a})}$, counting both a and b as expressed in calibers of the gun.

189. We shall now apply this formula in calculating the greatest velocity of shot discharged from all guns, in a series of all the lengths, differing by 2 calibers at a time, with the respective length of the charges that communicate the greatest velocity; the results, with the correspondent data, are arranged in the following table, which are all the same, whatever the caliber or diameter may be; the powder put in close to the plane or flat bottom of the bore, without a bag, and the diameter of the ball equal to the caliber of the gun, or as nearly so as possible, with very little or no windage.

Table of Charges for the greatest Velocities.

Length of the bore in calibers. b	Length of the powder in calibers. a	Quotient of $b \div a$, $\frac{b}{a}$	Wt. of powder in 100 parts of the wt. of the ball.	Greatest velocity of ball by each gun.
2	0·63	3·171	12	810
4	1·20	3·333	23	1122
6	1·72	3·488	33	1348
8	2·20	3·636	42	1529
10	2·64	3·788	50	1681
12	3·05	3·934	58	1813
14	3·43	4·082	65	1929
16	3·78	4·233	71	2033
18	4·11	4·380	78	2127
20	4·42	4·525	84	2213
22	4·71	4·671	90	2292
24	4·99	4·810	95	2366
26	5·25	4·952	100	2434
28	5·50	5·091	105	2498
30	5·73	5·235	109	2558
32	5·96	5·369	113	2614
34	6·17	5·510	117	2668
36	6·37	5·651	121	2719
38	6·56	5·793	125	2767
40	6·75	5·926	128	2813
42	6·93	6·061	132	2857
44	7·10	6·197	135	2899
46	7·27	6·328	138	2939
48	7·43	6·460	141	2978
50	7·58	6·596	143	3015
52	7·72	6·736	146	3051
54	7·86	6·870	149	3085
56	8·00	7·000	152	3118
58	8·13	7·134	155	3150
60	8·26	7·264	157	3181

190. In this table, the first column contains the different lengths of the guns or mortars, expressed in the number of calibers, or how many times the diameter is contained in the length, differing by 2 diameters each time, and extended from the shortest to the very longest piece, so as to include

all the lengths and diameters of pieces that can be used. In like manner, the 2nd column shows all the correspondent lengths of the charge of pure powder, put in without any bag or wad, but set close up with the rammer, these being the charges to produce the greatest velocity of the ball in each gun, and expressed in calibers of the gun also; all those values of a, or length of the charge, being determined by the foregoing theorem $a^2 + (48 + b)\,a = 16b$, in art. 186, by using for the value of b the successive numbers 2, 4, 6, 8, &c, in the first column. The 3rd column exhibits the quotients of b divided by a, or the values of $\frac{b}{a}$, being the proportions of the length of the bore to the length of the charge. The 4th column shows, in so many 100th parts of the weight of the ball, what is the weight of the several charges of powder whose lengths are contained in the 2d column, to produce the greatest velocity in the ball. Thus, for the first piece, of only 2 calibers long, the weight of the charge to give the greatest velocity, is $\frac{12}{100}$, or about $\frac{1}{8}$ of the weight of the shot; for the piece of 10 calibers, the weight of the charge is $\frac{50}{100}$, or just $\frac{1}{2}$ the weight of the ball; for 26 calibers bore, the weight of the powder is 100, or just the weight of the ball; and so on to the last, of 60 calibers bore, where the weight of the powder is $\frac{157}{100}$ of that of the ball, or more than $1\frac{1}{2}$. And this column of the weights of the powder is made out from that of the lengths of the charges, in the following manner. In art. 184 it is shown that the weight of the ball is $\frac{8}{3}{}^{6}cd^3$ ounces, and that of the powder $\cdot 54225acd^2$, which two quantities are to each other in proportion as d to $\cdot 19a$ or $\frac{1}{5}$ of a nearly, where d denotes the diameter or 1 caliber, a the calibers in the length of powder; therefore the values of a, in the 2d column, multiplied by $\cdot 19$, or divided by $5\cdot 26$, will give the weight of the powder in 100th parts of the ball, as they are arranged in the 4th column. Lastly, the 5th column contains the several greatest velocities by the same guns, with the foregoing charges of powder, as computed from the formula $5875\sqrt{(\frac{a}{16+a}} \times$

log. of $\frac{b}{a}$), where a and b are expressed in calibers of the gun, as they stand in the first and second columns of the last table. So that this table and the rules are general for all calibers or diameters of guns, as well as all lengths; and from which the practical artillerist may derive the greatest advantage. If he wish to know what may be the best charge for any gun he wants to employ, knowing only how many calibers it is in length, with the weight of the ball; by looking in the table, he sees at once both the length of the charge of powder, and its weight, as well as the degree of velocity it will communicate. If the length, in any case, fall between any two adjacent numbers in the first column, it is then only necessary to take the like medium between the correspondent numbers in the columns of powder and velocity. If a less degree of velocity be necessary, the correspondent charge of powder may be found, by taking it in proportion to the square of the velocity; viz, by saying, as the square of the tabular velocity is to the tabular charge of powder, so is the square of the proposed velocity, to the charge desired: and thus in many cases a needless waste of powder and velocity may be prevented. In fine, it may be with truth remarked that this table, and the foregoing rules, being deduced from very numerous and laborious experiments, may be depended on for practice, as far more exact and satisfactory than any thing of the kind that has been before given for such purposes.

TRACT XXXVIII.

MISCELLANEOUS PRACTICAL PROBLEMS, &c, ILLUSTRATING SOME OF THE FOREGOING PRINCIPLES.

PROBLEM I.

It is required to find the Diameter of a Circular Parachute, by means of which a man of 150*lb weight may descend to the earth, from a Balloon at a height in the air, with the Velocity of only* 10 *feet in a second of time, being the Velocity acquired by a body freely descending through a space of only* 1 *foot* $6\frac{1}{4}$ *inches, or of a man jumping down from a height of* $18\frac{3}{4}$ *inches: the Parachute being made of such materials and thickness, that a circle of it of* 50 *feet diameter, weighs only* 150*lb, and so in proportion more or less according to the area of the circle.*

If a falling body descend with a uniform velocity, it must necessarily meet with a resistance, from the medium it descends in, equal to the whole weight that descends. Let x denote the diameter of the parachute, and $a = \cdot 7854$; then ax^2 will be its area, and as $50^2 : x^2 :: 150 : \frac{3}{50}x^2$ the weight of the same, to which adding 150lb, the man's weight, the sum $\frac{3}{50}x^2 + 150$ will be the whole descending weight. Again, in the table of resistances, at pa. 189, Tract 36, art. 42, we find that a circle of $\frac{2}{9}$ of a square foot area, moving with 10 feet velocity, meets with a resistance of ·57 ounces=·0475lb; and the resistances, with the same velocity, being as the surfaces, therefore as $\frac{2}{9} : \cdot 0475 :: ax^2 : \cdot 21375ax^2 = \cdot 16788x^2$ the resistance of the air to the parachute, to which the descending weight must be equal; that is, $\cdot 16788x^2 = \frac{3}{50}x^2 + 150$; hence $\cdot 10788x^2 = 150$, or $x^2 = 1390\cdot 5$, and hence $x = 37\frac{2}{7}$ feet, the diameter of the parachute required.

PROBLEM II.

To determine the Effects of Pile-Engines.

The form of the pile-engine, as used by the ancients, is not known. Many have been invented and described by the moderns. Among all these, that appears to be the best which was invented by M. Vauloue, as described by Dr. Desaguliers, and was used at piling the foundations at building Westminster Bridge. Its chief properties are, that the ram or weight be raised with the least expence of force, or with the fewest men; that it fall freely from its greatest height; and that, having fallen, it is presently laid hold of by the forceps, and so raised up to its height again. By which means, in the shortest time, and with the fewest men, or the least force, the most piles can be driven to the greatest depth.

Belidor has given some theory as to the effect of the pile-engine; but it appears to be founded on an erroneous principle: he deduces it from the laws of the collision of bodies. But who does not perceive that the rules of collision suppose a free motion and a non-resisting medium? It cannot therefore be applied in the present case, where a very great resistance is opposed to the pile by the ground. We shall therefore here endeavour to explain another theory of this machine.

Since the percussion of the weight acts on the pile during the whole time the pile is penetrating and sinking in the earth, by each blow of the ram, during which time its whole force is spent; it is manifest that the effect of the blow is of that nature, which requires its force to be estimated by the square of the velocity. But the square of the velocity acquired by the fall of the ram, is as the height it falls from; therefore the force of any blow will be as the height fallen through. But it is also more or less in proportion to the weight of the ram; consequently the effect or force of each blow must be directly in the compound ratio of both, viz, as aw, where w denotes the weight, and a the altitude it falls

from; or it will be simply as the altitude a, when the weight w is constant.

Again, the force of the blow is opposed by the mass of the pile, and by the consistence of the earth penetrated by the point of the pile, and also by the friction of the earth against the sides of the pile that have penetrated below the surface. Consequently the effect of the blow, or the depth penetrated by the pile, will be inversely in the compound ratio of these three, viz, inversely as mtf, where m denotes the mass of the pile, t the tenacity or cohesion of the earth, and f the friction of the surface penetrated in the earth. But, in the same soil and with the same pile, m and t are both constant, in which case the depth of penetration will be inversely only as f the friction. On all accounts then the penetration will be as $\frac{aw}{mtf}$, or simply as $\frac{a}{f}$ only, for the same weight and pile and soil.

To determine the depth sunk by the Pile, at each stroke of the ram.

After a few strokes, so as to give the pile a little hold in the ground, to make it stand firmly, the blows of the ram may be considered as commencing, and causing the pile to sink a little at every stroke, by which small successive sinkings of the pile, the space the ram falls through will be successively increased by these small accessions, and the force of the successive blows proportionally increased. But these, on the other hand, are resisted and opposed by the friction of the part of the pile which has been sunk before, and which also sinks at each stroke; and as the quantities of these rubbing surfaces increase in a greater ratio to each other, than the heights fallen through, that is, the resisting forces increasing faster than the impelling forces, it is manifest that the depths successively sunk by the blows must gradually decrease by little and little every time; which is also found to be quite conformable to experience. Thus then the successive sinkings will proceed gradually diminishing, till they become so small as to be almost imperceptible.

Now it was found above that $\frac{a}{f}$ is as the penetration by any blow of the ram, by the same pile in the same soil, that is, as the height fallen directly, and as the resistance or friction in the earth inversely. Let a' denote any other and greater height, by an after stroke, and f' its friction; also p the penetration by the former blow, and p' that by the latter, which must be the smaller: then, by the foregoing principle, $\frac{a}{f} : \frac{a'}{f'} :: p : p'$; hence $a : a' :: pf : p'f'$, which is a general theorem.

But now, with respect to the quantity of friction from any blow, though it be not known from experiment that the friction is exactly proportional to the rubbing surface, there is great reason to believe that it must be at least very nearly so: there is also equal reason to conclude that the effect or resistance from that rubbing surface must be nearly or exactly as the length of space it moves over, that is, by the penetration of the pile by any blow. Now, if d denote the depth of the pile in the ground before any new blow is struck by the ram, and b the depth or penetration produced by the blow, then the length of the rubbing surface will be $d + \frac{1}{2}b$; for, the length of the rubbing surface is only d at the beginning of the motion, and it is $d + b$ at the end of it, the medium of the two, or $d + \frac{1}{2}b$, is therefore the due length of the surface, and the space or depth it moves over is b; therefore the whole resistance from the friction is $(d + \frac{1}{2}b)b$. If d' then denote any other depth of the pile in the earth, and b' the next penetration, then $(d' + \frac{1}{2}b')b'$ will be its friction. Substituting now b for p, and b' for p', also $d + \frac{1}{2}b$ for f, and $d' + \frac{1}{2}b'$ for f', in the general theorem $a : a' :: pf : p'f'$, it becomes $a : a' :: (d + \frac{1}{2}b)b : (d' + \frac{1}{2}b')b'$, for the general relation between the heights fallen and the resistance and penetration.

This theorem will very conveniently give the series of effects, or successive sinkings of the piles, by the blows of the ram. Thus, after the pile has been properly fixed, or indeed driven to any depth in the earth, denoted by d, then to give

a blow, the ram falls from the height $a + d$, and thereby sinks the pile the space b suppose; hence, for the next stroke, the fall will be $a + d + b = a'$ in the theorem above, and $d' + \frac{1}{2}b' = d + b + \frac{1}{2}b'$, the next penetration or sinking being b'; theref. $a + d : a + d + b :: (d + \frac{1}{2}b)b : (d + b + \frac{1}{2}b')b'$, a proportion which gives the quadratic equa. $b'^2 + 2b'(d + b) = \frac{a+d+b}{a+d} \times (2d + b)b$, the root of which is $b' = -(d + b) + \sqrt{[(d + b)^2 + \frac{a+d+b}{a+d} \times (2d + b)b]} = \frac{a + d + b}{a + d} \times \frac{d + \frac{1}{2}b}{d + b}b$ nearly, or indeed $= \frac{d+\frac{1}{2}b}{d+b}b$ nearly, because b is small in comparison with $a + d$.

Now, for an example in numbers, suppose $a = 5$ feet $=$ 60 inches, $d = 10$, $b = 3$, that is $a = 60$ the height of the ram above the top of the pile before this enters the ground; $d = 10$, after being fixed in the ground; and $b = 3$ the sinking by the next blow: then $\frac{d+\frac{1}{2}b}{d+b}b = \frac{11.5}{13} \times 3 = 2.65 = b'$ the 2d stroke. Next, substituting $d + b$ for d, and b' for b, the same theorem gives 2·48 for the next sinking, or the next value of b'. And so on continually, by which means the series of the successive corresponding values of the letters will be as in the margin, the last column showing the several successive sinkings of the pile by the repeated strokes of the ram.

Specimen of the Series of the Successive values of d, b, b'.

d	b	b'
10	3	2·65
13	2·65	2·48
15·65	2·49	2·32
18·14	2·32	2·19
20·46	2·19	2·08
&c.		

Scholium. Thus then it appears, that the effect of any operation of pile-driving may be determined. It is manifest also that the greater a is, or the higher the top of the machine is where the ram falls from, above the top of the pile at first, the greater will be every stroke of the ram, and consequently the fewer the strokes requisite to drive the pile to the requisite depth. But then every stroke will take a longer time, as the ram will be both longer in falling and longer in rais-

ing: so that it may be a question whether, on the whole, the business may be effected in the less time by a greater height of the machine, or whether there be any limit to the height, so as to produce the greatest effect in a given time.

To answer this question, let x denote the indeterminate height from which any weight w is to fall, z the time of raising it after a fall, which time is supposed to be as the height x to which it is raised, also m the given time of producing a proposed effect; then $\frac{1}{4}\sqrt{x}$ = the time of the weight falling; therefore $\frac{1}{4}\sqrt{x}+z$ = the whole time of one stroke; conseq. $\frac{m}{\frac{1}{4}\sqrt{x}+z}$ or $\frac{4m}{\sqrt{x}+4z}$ is the number of strokes made in the given time m, and hence $\frac{4mzw}{\sqrt{x}+4z}$ = the whole force or effect in the time m. Now this effect or fraction increases continually as x increases, because the numerator increases faster than the denominator, since the former increases as x, while, in the latter, though the one term z increases as x, yet the other term $\sqrt{x}$ only increases as the root of x. So that, on the whole, it appears that the effect, in any given time, increases more and more as the height is increased.

PROBLEM III.

To determine how far a man, who pushes with the force of 100*lb, can thrust a sponge into a piece of ordnance, whose diameter is* 5 *inches, and length* 10 *feet, when the barometer stands at* 30 *inches: the vent, or touch-hole, being stopped, and the sponge having no windage, that is, fitting the bore quite close?*

A column of quicksilver 30 inches high, and 5 in diameter, is $5^2 \times 30 \times \cdot 7854 = 589\cdot05$ inches; which, at 8·102 oz each inch, weighs 4772·48 oz, or 298·28lb, which is the pressure of the atmosphere alone, being equal to the elasticity of the air in its natural state; to this adding the 100lb, gives 398·28lb, the whole external pressure. Then, as the spaces which a quantity of air possesses, under different pressures, are in the reciprocal ratio of those pressures, it

will be, as 398·28 : 298·28 : : 10 feet or 120 inches : 90 inches nearly, the space occupied by the air; theref. 120—90=30 inches, is the distance sought.

PROBLEM IV.

To assign the Cause of the Deflection of Military Projectiles.

It having been surmized that, in the practice of artillery, the deflection of the shot in its flight, to the right or left, from the line or direction the gun is laid in, chiefly arises from the motion of the gun during the time the shot is passing out of the piece: it is required then to determine what space an 18-pounder will recoil or fly back, while the shot is passing out of the gun; supposing its weight to be 4800lb, that of the carriage 2400lb, the quantity of powder 8lb, the length of the cylinder 108 inches, that of the charge 13 inches, and the diameter of the bore 5·13 inches; supposing also that the resistance from the friction between the platform and carriage is equal to 3600lb?

It is well known that confined gunpowder, when fired, immediately changes in a great measure into an elastic air, which endeavours to expand in all directions. Now, in the question, the action of this fluid is exerted equally on the bottom of the bore of the gun and on the ball, during the passage of the latter through the cylinder; the two bodies therefore move in opposite directions, with velocities which are at all times in the inverse ratio of the quantities of matter moved. Now let x be the space through which the gun recoils; then, as the charge occupies 13 inches of the barrel, and the semidiameter of the barrel is 2·565, the space moved through by the ball when it quits the piece, is $108 - 13 - 2{\cdot}565 - x = 92{\cdot}435 - x$: and as the elastic fluid expands in both directions, the quantity which advances towards the muzzle, is to that which retreats from it, as $92{\cdot}435 - x$ to x: consequently $\frac{8x}{92{\cdot}435}$ and $\frac{92{\cdot}435 - x}{92{\cdot}435} \times 8$ are the quantities of the powder which move, the former with the gun, and the latter

with the ball; besides these, the weight of ball that moves forwards being 18lb, and of the weights and resistance backwards 4800 + 2400 + 3600 = 10800lb, hence the whole weights moved in the two directions are $10800 + \frac{8x}{92\cdot435}$ and $18 + \frac{92\cdot435 - x}{92\cdot435} \times 8$, or $\frac{998298 + 8x}{92\cdot435}$ and $\frac{2403\cdot31 - 8x}{92\cdot435}$, or as the numerators of these only. But when the time and moving force are given, or the same, then the spaces are inversely as the quantities of matter; therefore $x : 92\cdot435 - x :: 2403\cdot31 - 8x : 998298 + 8x$, or by composition, $x : 92\cdot435 :: 2403\cdot31 - 8x : 1000701\cdot31$, and by div. $x : 1 :: 2403\cdot31 - 8x : 10826$, theref. $10826x = 2403\cdot31 - 8x$, or $10834x = 2403\cdot31$, and hence $x = \cdot2218$ inch $= \frac{2}{9}$ of an inch nearly, or the recoil of the gun is less than a quarter of an inch.

Hence it may be concluded, that so small a recoil, straight backwards, can have no effect in causing the ball to deviate from the pointed line of direction: and that it is very probable we are to seek for the cause of this effect in the ball striking or rubbing against the sides of the bore, in its passage through it, especially near the exit at the muzzle; by which it must happen, that if the ball strike against the right side, the ball will deviate to the left; if it strike on the left side, it must deviate to the right; if it strike against the under side, it must throw the ball upwards, and make it to range farther; but if it strike against the upper side, it must beat the ball downwards, and cause a shorter range: all which irregularities are found to take place, especially in guns that have much windage, or which have the balls too small for the bore.

PROBLEM V.

A Ball of Lead, of 4 inches diameter, being dropped from the top of a tower, of 65 yards high, falls into a cistern full of water at the bottom of the tower, of $20\frac{1}{4}$ yards deep: it is required to determine the times of falling, both to the surface and to the bottom of the water.

The fall in air is 195 feet, and in water $60\frac{3}{4}$ feet. By the common rules of descent, as $\sqrt{16} : \sqrt{195} :: 1'' : \frac{1}{4}\sqrt{195} = 3{\cdot}49$ seconds, the time of descending in air. And as $\sqrt{16} : \sqrt{195} :: 32 : 8\sqrt{195} = 111{\cdot}71$ feet, the velocity at the end of that time, or with which the ball enters the water.

Again, by prob. 22 of vol. 2, art. 2 of the Course, the space $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a-e^2}{a-v^2}$, or rather $\frac{1}{2b} \times$ hyp. log. of $\frac{e^2-a}{v^2-a}$ (the velocity being decreasing, and e^2 greater than a) $= \frac{m}{2b} \times$ com. log. of $\frac{e^2-a}{v^2-a}$, where $N = 11325$ the density of lead, $n = 1000$ that of water, $a = \frac{256d(N-n)}{3n}$, $b = \frac{3n}{8dN}$, $e = 111{\cdot}71$ the velocity at entering the water, and v the velocity at any time afterwards, also d the diameter of the ball $= 4$ inches, and $m = 2{\cdot}302585$ the hyp. log. of 10.

Hence then $N = 11325$, $n = 1000$, $N - n = 10325$, $d = \frac{4}{12} = \frac{1}{3}$; then $a = \frac{256d(N-n)}{3n} = \frac{256 \,.\, 10325}{9000} = 293\frac{1}{4}$, and $b = \frac{3n}{8dN} = \frac{9n}{8N} = \frac{9000}{90600} = \frac{15}{151} = \frac{1}{10}$ nearly. Also $e = 111{\cdot}71$; therefore $s = 60\frac{3}{4} = \frac{m}{2b} \times$ log. of $\frac{e^2-a}{v^2-a} = 5m \times$ log. $\frac{e^2-a}{v^2-a}$. This theorem will give s when v is given, and by reverting it will give v in terms of s in the following manner.

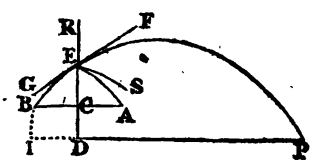
Dividing by $5m$ gives $\frac{s}{5m} =$ log. of $\frac{e^2-a}{v^2-a} = ns$, by putting $n = \frac{1}{5m}$; therefore, the natural number is $10^{ns} = \frac{e^2-a}{v^2-a}$; hence $v^2 - a = \frac{e^2-a}{10^{ns}}$, and $v = \sqrt{(a + \frac{e^2-a}{10^{ns}})}$, which, by substituting the numbers above mentioned for the letters,

gives $v = 17{\cdot}134$ for the last velocity, when the space $s = 60\frac{3}{4}$, or when the ball arrives at the bottom of the water.

But now to find the time of passing through the water, putting t = any time in motion, and s and v the corresponding space and velocity, the general theorem for variable forces gives $\dot{t} = \frac{\dot{s}}{v}$. But the above general value of s being $\frac{1}{2b} \times$ hyp. log. $\frac{e^2-a}{v^2-a}$ or $5 \times$ hyp. log. $\frac{e^2-a}{v^2-a}$, therefore its fluxion $\dot{s} = \frac{-10v\dot{v}}{v^2-a}$, consequently, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{-10\dot{v}}{v^2-a}$, the correct fluent of which is $\frac{5}{\sqrt{a}} \times$ hyp. log. $\left(\frac{e-\sqrt{a}}{e+\sqrt{a}} \times \frac{v+\sqrt{a}}{v-\sqrt{a}}\right) = t$ the time, which when $v = 17{\cdot}134$, or $s = 60\frac{3}{4}$, gives $2{\cdot}6542$ seconds, for the time of descent through the water.

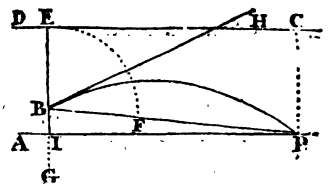
PROBLEM VI.

A person standing at the distance of 10 *feet from the bottom of a wall, which is supposed perfectly smooth and hard, desires to know in what direction he must throw an elastic ball against it, with a velocity of* 80 *feet per second, so that, after reflection from the wall, it may fall at the greatest distance possible from the bottom, on the horizontal plane, which is* $2\frac{1}{2}$ *feet below the hand discharging the ball?*



In the annexed figure let DR be the wall against which the ball is thrown, from the point A, in such a direction, that it shall describe the parabolic curve AE before striking the wall, and afterwards be so reflected as to describe the curve EP. Now if ES be the tangent at the point E, to the curve AE described before the reflection, and EF the tangent at the same point to the curve which the ball will describe after reflection, then will the angle REF be = CES; and if the curve PE be produced, so as to have GF for its tangent, it will meet AC produced in B, making BC = AC, and the curve AE will be similar and equal

to the portion BE of the parabola BEP, but turned the contrary way. Conceiving either the two curves AE and EP, or the continued curve BEP, to be described by a projectile in its motion, it is manifest that, whether the greater portion of the curve be described before or after the ball reaches the wall DR, will depend on its initial velocity, and on the distance AC or BC, and on the angle of projection. The problem then is now reduced to this, viz, To find the angle at which a ball shall be projected from B, with a given impetus, so that the distance DP, at which it falls, from the given point D, on the plane DP, parallel to the horizon, shall be a maximum.



Now this problem may be constructed in the following manner: From any point E in the horizontal line DC, let fall the indefinite perp. EG, on which set off EB = the impetus corresponding to the given velocity, and BI = $2\frac{1}{2}$ the distance of the horizontal plane below the point of projection; also, through I draw AP parallel to DC. From the point B set off BP = BE + EI, and bisect the angle EBP by the line BH: then will BH be the required direction of the ball, and IP the maximum range on the plane AP.

For, since the ball moves from the point B, with the velocity acquired by falling through EB, it is manifest, from p. 156 vol. 2 of the Math. Course, that DC is the directrix of the parabola described by the ball. And since both B and P are points in the curve, each of them must, from the nature of the parabola, be as far from the focus as it is from the directrix; therefore B and P will be the greatest distance from each other when the focus F is directly between them, that is, when BP = BE + CP. And when BP is a maximum, since BI is constant, it is obvious that IP is a maximum also. Further, the angle FBH being = EBH, the line BH is a tangent to the parabola at the point B, and consequently it is the direction necessary to give the range IP.

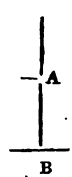
Cor. 1. When B coincides with I, IP will be = BP = BE + EI = 2EI, and the angle EBH will be 45°; as is also manifest from the common modes of investigation.

Cor. 2. When the impetus corresponding to the initial velocity of the ball is very great compared with AC or BC (fig. 1), then the part AE of the curve will very nearly coincide with its tangent, and the direction and velocity at A may be accounted the same as those at E without any sensible error. In this case too the impetus BE (fig. 2) will be very great compared with BI, and consequently, B and I nearly coinciding, the angle EBH will differ but little from 45°.

Calcul. From the foregoing construction the calculation will be very easy. Thus, the first velocity being 80 feet = v, then (vol. 2 pa. 156 of the Course) $\frac{v^2}{4g} = \frac{80 \times 80}{64\frac{1}{3}} = 99{\cdot}48186$ = BE the impetus; hence EI = FP = 101·98186, and BP = BE + EI = 201·46372. Now, in the right-angled triangle BIP, the sides BI and BP are known, hence IP = 201·4482, and the angle IBP = 89° 17′ 20″: half the suppl. of this angle is 45° 21′ 20″ = EBH. And, in fig. 1, IP − ID = 201·4482 − 10 = 191·4482 = DP, the distance the ball falls from the wall after reflection.

PROBLEM VII.

From what height above the given point A *must an elastic ball be suffered to descend freely by gravity, so that, after striking the hard plane at* B, *it may be reflected back again, to the point* A, *in the least time possible from the instant of dropping it?*



Let C be the point required; and put AC = x, and AB = a; then $\frac{1}{4}\sqrt{CB} = \frac{1}{4}\sqrt{(a+x)}$ is the time in CB, and $\frac{1}{4}\sqrt{CA} = \frac{1}{4}\sqrt{x}$ is the time in CA; therefore $\frac{1}{4}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$ is the time down AB, and the time of rising from B to A again: hence the whole time of falling through CB and returning to A, is $\frac{1}{2}\sqrt{(a+x)} - \frac{1}{4}\sqrt{x}$; which must be a min. or $2\sqrt{(a+x)} - \sqrt{x}$ a minimum,

in fluxions $\frac{\dot{x}}{\sqrt{(a+x)}} - \frac{\dot{x}}{2\sqrt{x}} = 0$, and hence $x = \frac{1}{3}a$, that is, AC $= \frac{1}{3}$AB, gives the point C fallen from.

PROBLEM VIII.

A cylinder of wood is depressed in water till its top is just level with the surface, and then is suffered to ascend; it is required to determine the greatest altitude to which it will rise, and the other circumstances of its motions.

Let $a =$ the length, and b the area or base of the cylinder, m its specific gravity, that of water being 1, also $a - x$ any variable height through which the cylinder has ascended, or x being the part still immersed in the water. Then bx is the mass and force of the water upwards to raise the cylinder; and $a \times b \times m = abm$ is the weight of the cylinder opposing its ascent; therefore the motive force to raise the cylinder is $bx - abm$; also, the mass of the cylinder being abm, and that of the displaced water bx, the whole matter in motion is $bx + abm$; by which dividing the motive force, we have $\frac{bx - abm}{bx + abm} = \frac{x - am}{x + am} = f$ the accelerating force. Then the well known theorem $v\dot{v} = 32f \cdot - \dot{x}$, gives $v\dot{v} = 32\dot{x} \cdot \frac{am - x}{am + x}$, v being the velocity; and the correct fluent is $v^2 = 64\ (a - x - 2am \times$ hyp. log. of $\frac{am + a}{am + x})$ and hence $v = 8\sqrt{(a - x - 2am \times \text{h.l.} \frac{am + a}{am + x})}$ the general state of the velocity when the part x is immersed, or when the part $a - x$ is out of the water.

Now when the velocity arrives at its greatest state, by the opposite forces bx and abm becoming equal, then $x = am$, or $1 : m :: a : x$, that is, the whole length is to the part immersed, as the specific gravity of the fluid is to that of the cylinder. And if the latter be equal to half the former, which is nearly the case of fir timber, then $x = \frac{1}{2}a$ when the velocity is at the greatest. And the quantity of the greatest velocity is then equal to

7·786 feet per second nearly, taking 10 feet for the length of the cylinder.

After this state, the resistance gradually increasing more and more over the urging force, the velocity always decreases till it quite ceases, and the body becomes for an instant stationary. In that case the above expression for the velocity v becomes equal to 0, which consequently gives $a - x = 2am \times \text{h.l.} \frac{am + a}{am + x}$ for the part out of the water when the motion ceases. Or if $m = \frac{1}{2}$ as before, and the length of the cylinder be $a = 10$ feet for instance, the last equation becomes $10 - x = 10 \times \text{h.l.} \frac{15}{5 + x}$, and the root of this equation is $x = 1\frac{1}{4}$ very nearly, or $8\frac{3}{4}$ feet of the cylinder is out of the water when the upward motion ceases.

After the cylinder has arrived at its greatest height $8\frac{3}{4}$, where the upward motion ceases, the cylinder descends again to the same depth as at first, after which it again returns ascending as before; and so on, continually playing up and down to the same highest and lowest points, like the vibrations of a pendulum, the motion ceasing in both cases in a similar manner at the extreme points, then returning, it gradually accelerates till arriving at the middle point, where it is the greatest, then gradually retarding all the way to the next extremity of the vibration, thus making all the vibrations in equal times, to the same extent between the highest and lowest points, except that, by the small tenacity and friction &c, of the water against the sides of the cylinder, it will be gradually and slowly retarded in its motion, and the extent of the vibrations decrease till at length the cylinder, like the pendulum, come to rest in the middle point of its vibrations, where it naturally floats in its quiescent state, with the part am or half its length above the water.

PROBLEM IX.

Required to determine the quantity of matter in a sphere, the density varying as the nth power of the distance from the centre?

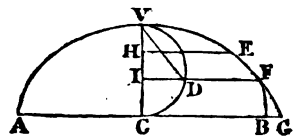
Let r denote the radius of the sphere, d the density at its surface, $a = 3{\cdot}1416$ the area of a circle whose radius is 1, and x any distance from the centre. Then $4ax^2$ will be the surface of a sphere whose radius is x, which may be considered by expansion as generating the magnitude of the solid; therefore $4ax^2\dot{x}$ will be the fluxion of the magnitude; but as $r^n : x^n :: d : \frac{dx^n}{r^n}$ the density at the distance x, therefore $4ax^2\dot{x} \times \frac{dx^n}{r^n} = \frac{4adx^{n+2}}{r^n}\dot{x}$ = the fluxion of the mass, the fluent of which $\frac{4adx^{n+3}}{(n+3)r^n}$, when $x = r$, is $\frac{4adr^3}{n+3}$, the quantity of the matter in the whole sphere.

Corol. 1. The magnitude of a sphere whose radius is r, being $\frac{4}{3}ar^3$, which call m; then the mass or solid content will be $\frac{3d}{n+3} \times m$, and the mean density is $\frac{3d}{n+3}$.

Corol. 2. It having been computed, from actual experiments, that the medium density of the whole mass of the earth is about $\frac{2}{5}$ times the density d at the surface, we can now determine what is the exponent of the decreasing ratio of the density from the centre to the circumference, supposing it to decrease by a regular law, viz, as x^n; for then it will be $\frac{2}{5}d = \frac{3d}{n+3}$, and hence $n = -\frac{4}{3}$. So that, in this case, the law of decrease is as $x^{-\frac{4}{3}}$, or as $\frac{1}{x^{\frac{4}{3}}}$, that is, inversely as the $\frac{4}{3}$ power of the distance from the centre.

PROBLEM X.

Required to determine where a body, moving down the convex side of a cycloid, will fly off and quit the curve.



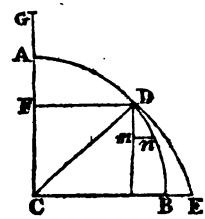
Let AVEB represent the cycloid, the properties of which may be seen at arts. 146 and 147 vol. 2 of the Course, and VDC its generating semicircle. Let E be the point where the motion commences, whence it moves along the curve, its velocity increasing both on the curve, and also in the horizontal direction DF, till it come to such a point, F suppose, that the velocity in the latter direction is become a constant quantity, then that will be the point where it will quit the cycloid, and afterwards describe a parabola FG, because the horizontal velocity in the latter curve is always the same constant quantity, by art. 76 vol. 2 of the Academy Course.

Put the diameter VC $= d$, VH $= a$, VI $= x$; then VD $= \sqrt{dx}$, and ID $= \sqrt{(dx - x^2)}$. Now the velocity in the curve at F, in descending down EF, being the same as by falling through HI or $x - a$, will be $= 8\sqrt{(x - a)}$; but this velocity in the curve at F, is to the horizontal velocity there, as VD to ID, because VD is parallel to the curve or to the tangent at F, that is $\sqrt{dx} : \sqrt{(dx - x^2)} :: 8\sqrt{(x - a)} : \frac{8\sqrt{(x-a)} \times \sqrt{(d-x)}}{\sqrt{d}}$, which is the horizontal velocity at F, where the body is supposed to have that velocity a constant quantity; therefore also $\sqrt{(x - a)} \times \sqrt{(d - x)}$, as well as $(x - a) \times (d - x) = ax + dx - ad - x^2$ is a constant quantity, and also $ax + dx - x^2$; but the fluxion of a constant quantity is equal to nothing, that is $a\dot{x} + d\dot{x} - 2x\dot{x} = 0 = a + d - 2x$, and hence $x = \frac{1}{2}a + \frac{1}{2}d$ $=$ VI, the arithmetical mean between VH and VC.

If the motion should commence at V, then x or VI would be $= \frac{1}{2}d$, and I would be the centre of the semicircle.

PROBLEM XI.

If a body begin to move from A, *with a given velocity, along the quadrant of a circle* AB; *it is required to show at what point it will fly off from the curve.*



Let D denote the point where the body quits the circle ABD, and then describes the parabola DE. Draw the ordinate DF, and let GA be the height producing the velocity at A. Put GA $= a$, AC or CD $= r$, AF $= x$; then the velocity in the curve at D will be the same as that acquired by falling through GF or $a + x$, which is, as before, $8\sqrt{(a + x)}$; but the velocity in the curve is to the horizontal velocity as Dn to mn or as CD to CF by similar triangles, that is, as $r : r - x :: 8\sqrt{(x + a)} : 8\sqrt{(x + a)} \times \frac{r - x}{r}$, which is to be a constant quantity where the body leaves the circle, therefore also $(r - x)\sqrt{(x + a)}$ and $(r - x)^2 \times (x + a)$ a constant quantity; the fluxion of which made to vanish, gives $x = \frac{r - 2a}{3} =$ AF.

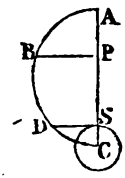
Hence, if $a = 0$, or the body only commence motion at A, then $x = \frac{1}{3}r$, or AF $= \frac{1}{3}$AC when it quits the circle at D. But if a or GA were $= \frac{1}{2}r$ or $\frac{1}{2}$AC, then $r - 2a = 0$, and the body would instantly quit the circle at the vertex A, and describe a parabola circumscribing it, and having the same vertex A.

PROBLEM XII.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ *feet, or* 193 *inches, in the first second of time.*

Put

r = CS the radius of the earth,
a = CA the dist. fallen from,
x = CP any variable distance,
v = the velocity at P,
t = time of falling there, and
g = $16\frac{1}{12}$, half the veloc. or force at S,
f = the force at the point P.



Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is considered as 1;

$\dot{t}v = -\dot{x}$, because x decreases; and

$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{4gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{4gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(4gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when r = 3965 miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} +$ arc to diameter a and vers. $a - x$); or the time of falling to any point P =

$\frac{1}{2r}\sqrt{\frac{a}{g}} \times$ (AB + BP). And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{AD + DS}{SC}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - $\sqrt{\frac{1}{4gr^2}} \times 2DS = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46′ 46″, which is the time of falling from the moon to the earth.

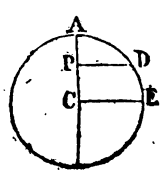
When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - - $\frac{1}{2r}\sqrt{\frac{a}{g}} \times ABDC = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{g}}$.

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XIII.

The force of attraction below the earth's surface being directly as the distance from the centre; it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth; abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.

Put $r =$ AC the radius of the earth,
$x =$ CP the dist. from the centre,
$v =$ the velocity at P,
$t =$ the time there,
$g = 16\frac{1}{12}$, half the force at A,
$f =$ the force at P.



Then CA : CP : : 1 : f; and the three equations are $rf = x$; and $v\dot{v} = -2gf\dot{x}$, and $\dot{t}v = -\dot{x}$. Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(2g \times \frac{r^2-x^2}{r})}$ $=$ PD$\sqrt{\frac{2g}{r}} =$ PD$\sqrt{\frac{2g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{(2g.\text{AC})}$ $=$ 25950 feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time, $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{(r^2-x^2)}}$, and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

When P arrives at C, the above becomes $t =$ $\sqrt{\frac{1}{2gr}} \times$ quadrant AE $= \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ seconds $= 21'\ 7''\frac{1}{4}$, for the time of falling to the centre C.

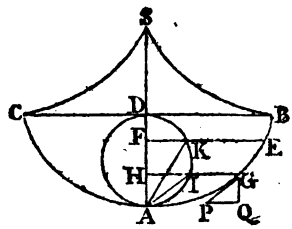
The time of falling to the centre is the same quantity $1{\cdot}5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For let n be any given distance from C at which the motion commences: then, by correction, $v =$ $\sqrt{[\frac{2g}{r}(n^2-x^2)]}$; and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{(n^2-x^2)}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1{\cdot}5708\sqrt{\frac{r}{2g}}$ for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A; then from B it will return again in the same manner, through C to A; and so vibrate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time

of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3{\cdot}1416\sqrt{\frac{r}{2g}} = 1^h\ 24'\ 29''$.

PROBLEM XIV.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let S be the point of suspension,

SA = the arc SB or SC the length of the pendulum,

CA = AB = SB or SC the semi-cycloid,

AD = DS the diameter of its generating circle, to which FKE, HIG are perpendiculars.

To any point G draw the tangent GP, also draw GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direct. GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of grav. : force in curve at E : : AD : AK; that is, the accelerative force in the curve, is as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI. So that the process and conclusions for the velocity and time of describing any arc in this case, will be the same as in the last problem, the nature of the forces being the same, viz, as the distance to be passed over to the lowest point A.

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c, is the same constant quantity

$1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{A}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$, and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{2g}}$, where l = AS = AB is the length of the pendulum, $g = 16\frac{1}{12}$ feet, or 193 inches, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, to the time of one vibration.

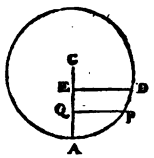
If the time of the whole vibration be 1 second, this equation arises, $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$, and hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. When g, for instance, is supposed to be $16\frac{1}{12}$ feet, or 193 inches, then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$ the length of a pendulum to vibrate seconds. Or if $l = 39\frac{1}{8}$, the length of the seconds pendulum for the latitude of London, then is $g = 4{\cdot}9348l = 193{\cdot}07$ inches $= 16\frac{107}{1200}$ feet, or nearly $16\frac{1}{12}$ feet, for the space descended by gravity in the first second of time in the latitude of London, also agreeing with experiment.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc, as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds in the very small arc of a circle.

PROBLEM XV.

To find the Velocity and Time of a heavy Body descending down the Arc of a Circle, or vibrating in the Arc by a Line fixed in the Centre.



Let D be the beginning of the descent, C the centre, and A the lowest point of the circle; draw DE and PQ perpendicular to AC. Then the velocity in P being the same as in Q by falling through EQ, it will be $v = 2\sqrt{(g \times \text{EQ})} = 8\sqrt{(a - x)}$, when $a = \text{AE}$, $x = \text{AQ}$.

But the flux. of the time $\dot{t}$ is $= \frac{-\dot{\text{AP}}}{v}$, and $\dot{\text{AP}} = \frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$ where r = the radius AC. Therefore $\dot{t} = \frac{r}{8} \times \frac{-\dot{x}}{\sqrt{(2rx - x^2)} \times \sqrt{(a - x)}}$ $= \frac{d}{16} \times \frac{-\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(d - x)}} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)} \times \sqrt{(1 - \frac{x}{d})}}$, where $d = 2r$ the diameter.

Or $\dot{t} = \frac{-\sqrt{d}}{16} \times \frac{\dot{x}}{\sqrt{(ax - x^2)}} (1 + \frac{x}{2d} + \frac{1 \,.\, 3x^2}{2 \,.\, 4d^2} + \frac{1 \,.\, 3 \,.\, 5x^3}{2 \,.\, 4 \,.\, 6d^3}$ &c), by developing $\sqrt{(1 - \frac{x}{d})}$ in a series.

But the fluent of $\frac{\dot{x}}{\sqrt{(ax - x^2)}}$ is $\frac{2}{a} \times$ arc to radius $\frac{1}{2}a$ and vers. x, or it is the arc whose rad. is 1 and vers. $\frac{2x}{a}$: which call A. And let the fluents of the succeeding terms, without the coefficients, be B, C, D, E, &c. Then will the flux. of any one, as $\dot{\text{Q}}$, at n distance from A, be $\dot{\text{Q}} = x^n\dot{\text{A}} = x\dot{\text{P}}$, which suppose also = the flux. of $b\text{P} - dx^{n-1}\sqrt{(ax - x^2)} = b\dot{\text{P}} - d(n - 1)\dot{x}x^{n-2}\sqrt{(ax - x^2)} - d\dot{x}x^{n-2} \times \frac{\frac{1}{2}ax - x^2}{\sqrt{(ax - x^2)}} = b\dot{\text{P}} - d\dot{x} \times \frac{(n - \frac{1}{2})ax^{n-1} - nx^n}{\sqrt{(ax - x^2)}} = b\dot{\text{P}} - d(n - \frac{1}{2})a\dot{\text{P}} + dnx\text{P}$.

Hence, by equating the coefficients of the like terms, $d = \frac{1}{n}$; $b = \frac{2n - 1}{2n}a$; and $\text{Q} = \frac{(2n - 1)a\text{P} - 2x^{n-1}\sqrt{(ax - x^2)}}{2n}$. Which being substituted, the fluential terms become $\frac{\sqrt{d}}{16} \times$ $(-\text{A} - \frac{1}{2d} \cdot \frac{a\text{A} - 2\sqrt{(ax - x^2)}}{2} - \frac{1 \,.\, 3}{2 \,.\, 4d^2} \cdot \frac{3a\text{B} - 2x\sqrt{(ax - x^2)}}{4} -$

$\frac{1.3.5}{2.4.6d^3}.\frac{5ac-2x^2\sqrt{(ax-x^2)}}{6}$ – &c). Or the same fluents will be found by art. 32, pa. 238, vol. 3 of the Course.

But when $x = a$, those terms become barely $\frac{3{\cdot}1416\sqrt{d}}{16} \times (-1 - \frac{1^2a}{2^2d} - \frac{1^2.3^2a^2}{2^2.4^2d^2} - \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3}$ – &c); which being subtracted, and x taken $= 0$, there arises for the whole time of descending down DA, or the corrected value of $t = \frac{3{\cdot}1416\sqrt{d}}{16} \times (1 + \frac{1^2a}{2^2d} + \frac{1^2.3^2a^2}{2^2.4^2d^2} + \frac{1^2.3^2.5^2a^3}{2^2.4^2.6^2d^3}$ + &c).

When the arc is small, as in the vibration of the pendulum of a clock, all the terms of the series may be omitted after the second, and then the time of a semi-vibration t is nearly $= \frac{1{\cdot}5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{a}{8r})$. And theref. the times of vibration of a pendulum, in different arcs, are as $8r + a$, or 8 times the radius added to the versed sine of the arc.

If D be the degrees of the pendulum's vibration, on each side of the lowest point of the small arc, the radius being r, the diameter d, and $3{\cdot}1416 = p$; then is the length of that arc $A = \frac{prD}{180} = \frac{pdD}{360}$. But the versed sine in terms of the arc is $a = \frac{A^2}{2r} - \frac{A^4}{24r^3}$ + &c $= \frac{A^2}{d} - \frac{A^4}{3d^3}$ + &c. Therefore $\frac{a}{d} = \frac{A^2}{d^2} - \frac{A^4}{3d^4}$ + &c $= \frac{p^2D^2}{360^2} - \frac{p^4D^4}{3.360^4}$ + &c, or only $= \frac{p^2D^2}{360^2}$ the first term, by rejecting all the rest of the terms on account of their smallness, or $\frac{a}{d} = \frac{a}{2r}$ nearly $= \frac{D2}{13131}$. This value then being substituted for $\frac{a}{d}$ or $\frac{a}{2r}$ in the last near value of the time, it becomes $t = \frac{1{\cdot}5708}{4}\sqrt{\frac{r}{2}} \times (1 + \frac{D2}{52524})$ nearly. And therefore the times of vibration in different small arcs, are as $52524 + D^2$, or as 52524 added to the square of the number of degrees in the arc.

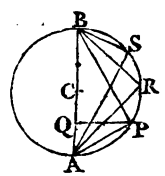
Hence it follows that the time lost in each second, by vibrating in a circle, instead of the cycloid, is $\frac{D^2}{52524}$; and consequently the time lost in a whole day of 24 hours, or $24 \times 60 \times 60$ seconds, is $\frac{5}{3}D^2$ nearly. In like manner, the seconds lost per day by vibrating in the arc of Δ degrees, is $\frac{5}{3}\Delta^2$.

Therefore, if the pendulum keep true time in one of these arcs, the seconds lost or gained per day, by vibrating in the other, will be $\frac{5}{3}(D^2 - \Delta^2)$. So, for example, if a pendulum measure true time in an arc of 3 degrees, it will lose $11\frac{2}{3}$ seconds a day by vibrating 4 degrees; and $26\frac{2}{3}$ seconds a day by vibrating 5 degrees; and so on.

And in like manner we might proceed for any other curve, as the ellipse, hyperbola, parabola, &c.

PROBLEM XVI.

To determine the Time of a Body descending down the Chord of a Circle.



Let C be the centre, AB the vertical diameter, AP any chord down which a body is to descend from P to A, and PQ perpendicular to AB. Now as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the time in PA : time in QA : : PA : QA : : BA : PA; but time in BA : time in QA : : $\sqrt{BA}$: $\sqrt{QA}$: : BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And in like manner the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and r = the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

Scholium. By comparing this with the results of the two preceding problems, it will appear that the times in the cy-

cloid, and in the arc of a circle, and in any chord of the circle, are respectively as the three quantities

$$1,\ 1 + \frac{a}{8r}\ \&c,\ \text{and}\ \frac{1}{\cdot 7854},$$

or nearly as the three quantities $1,\ 1 + \frac{a}{8r},\ 1\cdot 27324$; the first and last being constant, but the middle one, or the time in the circle, varying with the extent of the arc of vibration. Also the time in the cycloid is the least, but in the chord the greatest; for the greatest value of the series, in prob. 15, when $a = r$, or the arc AD is a quadrant, is $1\cdot 18014$; and in that case the proportion of the three times is as the numbers $1, 1\cdot 18014, 1\cdot 27324$. Moreover the time in the circle approaches to that in the cycloid, as the arc decreases, and they are very nearly equal when that arc is very small.

PROBLEM XVII.

To find the Time and Velocity of a Chain, consisting of very small links, descending from a smooth horizontal plane; the Chain being 100 *inches long, and* 1 *inch of it hanging off the Plane at the commencement of Motion.*

Put $a = 1$ inch, the length at the beginning;
$l = 100$ the whole length of the chain;
$x =$ any variable length off the plane.

Then x is the motive force to move the body,

and $\frac{x}{l} = f$ the accelerative force.

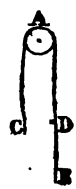
Hence $v\dot{v} = 2gf\dot{s} = 2g \times \frac{x}{l} \times \dot{x} = \frac{2gx\dot{x}}{l}$.

The fluents give $v^2 = \frac{2gx^2}{l}$. But $v = 0$ when $x = a$, theref. by correction, $v^2 = 2g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(2g \times \frac{x^2 - a^2}{l})}$ the velocity for any length x. And when the chain just quits the plain, $x = l$, and then the greatest velocity is $\sqrt{(2g \times \frac{l^2 - a^2}{l})} = \sqrt{(2 \times 193 \times \frac{100^2 - 1^2}{100}} = \sqrt{\frac{386 \times 9999}{100}} =$ 196·45902 inches, or 16·371585 feet per second.

Again $\dot{t}$ or $\frac{\dot{s}}{v} = \sqrt{\frac{l}{2g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$; the correct fluent of which is $t = \sqrt{\frac{l}{2g}} \times \log. \frac{x + \sqrt{x^2 - a^2}}{a}$, the time for any length x. And when $x = l = 100$, it is $t = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2{\cdot}69676$ seconds, the time when the last of the chain just quits the plane.

PROBLEM XVIII.

To find the Time and Velocity of a Chain, of very small Links, quitting a Pulley, by passing freely over it: the whole Length being 200 Inches, and the one End hanging 2 Inches below the other at the beginning.



Put $a = 2$, $l = 200$, and $x =$ BD any variable difference of the two parts AB, AC. Then $\frac{x}{l} = f$, and $v\dot{v}$ or $2gf\dot{s} = 2g.\frac{x}{l}.\frac{1}{2}\dot{x} = \frac{gx\dot{x}}{l}$.

Hence the correct fluent is $v^2 = g \times \frac{x^2 - a^2}{l}$, and $v = \sqrt{(g \times \frac{x^2-a^2}{l})}$, the general expression for the velocity. And when $x = l$, or when C arrives at A, it is $v = \sqrt{(g \times \frac{l^2-a^2}{l})} = \sqrt{(193 \times \frac{200^2-2^2}{200})} = \sqrt{(386 \times \frac{100^2-1^2}{100})} = \sqrt{\frac{386 \times 9999}{100}} = 196{\cdot}45902$ inches, or $16{\cdot}371585$ feet for the greatest velocity when the chain just quits the pulley.

Again, $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{\dot{x}}{2v} = \sqrt{\frac{l}{4g}} \times \frac{\dot{x}}{\sqrt{(x^2-a^2)}}$. And the correct fluent is $t = \sqrt{\frac{l}{4g}} \times \log. \frac{x + \sqrt{(x^2-a^2)}}{a}$, the general expression for the time. And when $x = l$, it becomes $t = \sqrt{\frac{l}{4g}} \times \log. \frac{l + \sqrt{(l^2-a^2)}}{a} = \sqrt{\frac{200}{772}} \times \log. \frac{200 + \sqrt{(200^2 - 2^2)}}{2} = \sqrt{\frac{100}{386}} \times \log. \frac{100 + \sqrt{9999}}{1} = 2{\cdot}69676$ seconds, the whole time when the chain just quits the pulley.

So that the velocity and time at quitting the pulley in this prob. and the plane in the last prob. are the same; the dis-

tance descended 99 being the same in both. For, though the weight l moved in this latter case, be double of what it was in the former, the moving force x is also double, because here the one end of the chain shortens as much as the other end lengthens, so that the space descended $\frac{1}{2}x$ is doubled, and becomes x; and hence the accelerative force $\frac{x}{l}$ or f is the same in both; and of course the velocity and time the same for the same distance descended.

PROBLEM XIX.

To find the Number of Vibrations made by two Weights, connected by a very fine Thread, passing freely over a Tack or a Pulley, while the less Weight is drawn up to it by the descent of the heavier Weight at the other End.

Suppose the motion to commence at equal distances below the pulley at B; and that the weights are 1 and 2 pounds.

Put a = AB, half the length of the thread;
$b = 39\frac{1}{8}$ inc. or $3\frac{25}{96}$ feet, the second's pend.
x = Bw = BW, any space passed over;
z = the number of vibrations.

Then $\frac{W-w}{W+w} = f = \frac{1}{3}$ is the accelerating force. And hence v or $\sqrt{4gfs} = \sqrt{4gfx}$, and $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{4gfx}}$. But, by the nature of pendulums, $\sqrt{(a \pm x)} : \sqrt{b} :: 1$ vibr. $: \sqrt{\frac{b}{a \pm x}}$ the vibrations per second made by either weight, namely, the longer or shorter, according as the upper or under sign is used, if the threads were to continue of that length for 1 second. Hence then, as

$$1'' : \dot{t} :: \sqrt{\frac{b}{a \pm x}} : \dot{z} = \dot{t}\sqrt{\frac{b}{a \pm x}} = \sqrt{\frac{b}{4gf}} \times \frac{\dot{x}}{\sqrt{(ax \pm x^2)}},$$

the fluxion of the number of vibrations.

Now when the upper sign + takes place, the fluent is

$$z = 2\sqrt{\frac{b}{4gf}} \times \text{l.}\ \frac{\sqrt{x} + \sqrt{(a+x)}}{\sqrt{a}} = \sqrt{\frac{b}{4gf}} \times \text{l.}\ \frac{a + 2x + 2\sqrt{(ax + x^2)}}{a}.$$

And when $x = a$, the same then becomes $z = \sqrt{\frac{b}{gf}} \times \log. 1 + \sqrt{2} = \sqrt{\frac{3b}{g}} \times \log. 1 + \sqrt{2} = \sqrt{\frac{117\frac{3}{8}}{193}} \times \log. 1 + \sqrt{2} =$ ·688511, the whole number of vibrations made by the descending weight.

But when the lower sign, or $-$, takes place, the fluent is $\sqrt{\frac{b}{4gf}} =$ arc to rad. 1 and vers. $\frac{2x}{a}$. Which, when $x = a$, gives $\frac{1}{2}p\sqrt{\frac{b}{gf}} = 3{\cdot}1416 \times \sqrt{\frac{8 \times 39\frac{1}{8}}{4 \times 193}} = \frac{3{\cdot}1416}{2} \times \sqrt{\frac{117\frac{3}{8}}{193}} =$ 1·227091, the whole number of vibrations made by the lesser or ascending weight.

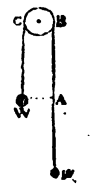
Schol. It is evident that the whole number of vibrations, in each case, is the same, whatever the length of the thread is. And that the greater number is to the less, as 1·5708 to the hyp. log. of $1 + \sqrt{2}$.

Farther, the number of vibrations performed in the same time t, by an invariable pendulum, constantly of the same length a, is $\sqrt{\frac{b}{gf}} =$ ·781190. For, the time of descending the space a, or the fluent of $\dot{t} = \frac{\dot{x}}{\sqrt{4gfx}}$, when $x = a$, is $t = \sqrt{\frac{a}{gf}}$. And, by the nature of pendulums, $\sqrt{a} : \sqrt{b} :: 1$ vibr. : $\sqrt{\frac{b}{a}}$ the number of vibrations performed in 1 second; hence $1'' : t :: \sqrt{\frac{b}{a}} : t\sqrt{\frac{b}{a}} = \sqrt{\frac{b}{gf}}$, the constant number of vibrations.

So that the three numbers of vibrations, namely, of the ascending, constant, and descending pendulums, are proportional to the numbers 1·5708, 1, and hyp. log. $1 + \sqrt{2}$, or as 1·5708, 1, and ·88137; whatever be the length of the thread.

PROBLEM XX.

To determine the Circumstances of the Ascent and Descent of two unequal Weights, suspended at the two Ends of a Thread passing over a Pulley: the Weight of the Thread and of the Pulley being considered in the Solution.



Let l = the whole length of the thread;
a = the weight of the same;
b = Aw the dif. of lengths at first;
d = W − w the dif. of the two weights;
c = a weight applied to the circumference, such as to be equal to its whole wt. and friction reduced to the circumference;
s = W+w+a+c the sum of the weights moved.

Then the weight of b is $\frac{ab}{l}$, and $d - \frac{ab}{l}$ is the moving force at first. But if x denote any variable space descended by W, or ascended by w, the difference of the lengths of the thread will be altered $2x$; so that the difference will then be $b-2x$, and its weight $\frac{b-2x}{l}a$; conseq. the motive force there will be $d - \frac{b-2x}{l}a = \frac{dl-ab+2ax}{l}$, and therefr. $\frac{dl-ab+2ax}{sl} = f$ the accelerating force there. Hence then $v\dot{v} = 2gf\dot{x} = 2g\dot{x} \times \frac{dl-ab+2ax}{sl}$; the fluents of which give $v^2 = 4g\dot{x} \times \frac{dl-ab+ax}{sl}$, or $v = 2\sqrt{\frac{ag}{sl}} \times \sqrt{(ex+x^2)}$ the general expression for the velocity, putting $e = \frac{dl-ab}{a}$. And when $x=b$, or W becomes as far below w as it was above it at the beginning, it is barely $v = 2\sqrt{\frac{bdg}{s}}$ for the velocity at that time. Also, when a, the weight of the thread, is nothing, the velocity is only $2\sqrt{\frac{dgx}{s}}$, as it ought.

Again, for the time, $\dot{t}$ or $\frac{\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{sl}{ag}} \times \frac{\dot{x}}{\sqrt{(ex+x^2)}}$; the fluents of which give $t = \sqrt{\frac{sl}{ag}} \times \log. \frac{\sqrt{x}+\sqrt{(e+x)}}{\sqrt{e}}$ the general expression for the time of descending any space x.

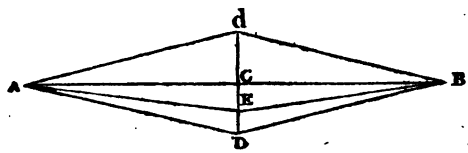
And if the radicals be expanded in a series, and the log. of it be taken, the same time will become

$$t = \sqrt{\frac{sx}{dg}} \times \sqrt{\frac{dl}{dl-ab}} \times \left(1 - \frac{x}{6e} + \frac{3x^2}{40e^2} \text{ \&c}\right).$$

Which therefore becomes barely $\sqrt{\frac{sx}{dg}}$ when a, the weight of the thread, is nothing, as it ought.

PROBLEM XXI.

To find the Velocity and Time of Vibration of a small Weight, fixed to the middle of a Line, or fine Thread void of Gravity, and stretched by a given Tension, the Extent of the Vibration being very small.



Let l = AC half the length of the thread;
a = CD the extent of the vibration;
x = CE any variable distance from C;
w = wt. of the small body fixed to the middle;
W = a wt. which, hung at each end of the thread, will be equal to the constant tension at each end, acting in the direction of the thread.

Now, by the nature of forces, AE : CE : : W the force in direction EA : the force in direction EC. Or, because AC is nearly = AE, the vibration being very small, taking AC instead of AE, it is AC : CE : W : $\frac{Wx}{l}$ the force in EC arising from the tension in EA. Which will be also the same for that in EB. Therefore the sum is $\frac{2Wx}{l}$ = the whole motive force in EC arising from the tensions on both sides. Consequently $\frac{2Wx}{lw} = f$ the accelerative force there. Hence the equation of the fluxions $v\dot{v}$ or $2gf\dot{s} = \frac{-4gWx\dot{x}}{lw}$; and the flus. $v^2 = -\frac{4gWx^2}{lw}$. But when $x = a$, this is $-\frac{4gWa^2}{lw}$, and should be = 0; theref. the correct fluents are $v^2 = 4gW \times \frac{a^2 - x^2}{lw}$, and hence $v = \sqrt{(4gW \times \frac{a^2-x^2}{lw})}$ the velocity of the little body w at any point E. And when $x = 0$, it is $v = 2a\sqrt{\frac{gW}{lw}}$ for the greatest velocity at the point C.

Now if we suppose $w = 1$ grain, $\text{W} = 5$lb troy, or 28800 grains, and $2l =$ AB $= 3$ feet; the velocity at C becomes $a\sqrt{\frac{8 \times 16\frac{1}{12} \times 28800}{3}} = 1111\frac{2}{3}a$. So that

if $a = \frac{1}{10}$ inc. the greatest veloc. is $9\frac{5}{19}$ ft. per sec.
if $a = 1$ inc. the greatest veloc. is $92\frac{37}{60}$ ft. per sec.
if $a = 6$ inc. the greatest veloc. is $555\frac{7}{10}$ ft. per sec.

To find the time t, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \frac{1}{2}\sqrt{\frac{lw}{\text{W}g}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}}$. Hence the correct fluent is $t = \frac{1}{2}\sqrt{\frac{wl}{\text{W}g}} \times$ arc to cosine $\frac{x}{a}$ and radius 1, for the time in DE. And when $x = 0$, the whole time in DC, or of half a vibration, is $\cdot 7854\sqrt{\frac{wl}{\text{W}g}}$; and conseq. the time of a whole vibration through Dd is $1\cdot 5708\sqrt{\frac{wl}{\text{W}g}}$.

Using the foregoing numbers, namely $w = 1$, $\text{W} = 28800$, and $2l = 3$ feet; this expression for the time gives $\frac{1111\frac{2}{3}}{3\cdot 1416} = 353\frac{5}{9}$, the number of vibrations per second. But if $w = 2$, there would be 250 vibrations per second; and if $w = 100$, there would be $35\frac{16}{45}$ vibrations per second.

PROBLEM XXII.

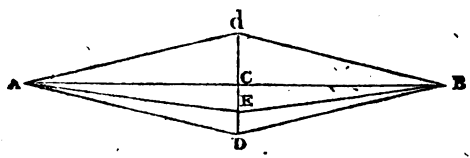
To determine the same as in the last Problem, when the Distance CD *bears some sensible proportion to the Length* AB; *the Tension of the Thread however being still supposed a constant Quantity.*

Using here the same notation as in the last problem, and taking the true variable length AE for AC, it is AE or EB : CE :: $2\text{W} : \frac{2\text{W}x}{\text{AE}} = \frac{2\text{W}x}{\sqrt{(l^2 + x^2)}}$ the whole motive force from the two equal tensions W in AE and EB; and theref. $\frac{2\text{W}}{w} \times \frac{x}{\sqrt{(l^2 + x^2)}} = f$ is the accelerative force at E. Theref. the fluxional equation is $v\dot{v}$ or $2gf\dot{s} = \frac{4\text{W}g}{w} \times \frac{-x\dot{x}}{\sqrt{(l^2 + x^2)}}$; and the fluents $v^2 = \frac{8\text{W}g}{w} \times -\sqrt{(l^2 + x^2)}$. But when $x = a$, these are $0 = \frac{8\text{W}g}{w} \times -\sqrt{(l^2 + a^2)}$; therefore the correct fluents are $v^2 = \frac{8\text{W}g}{w} \times$

$[\sqrt{(l^2 + a^2)} - \sqrt{(l^2 + x^2)}] = \frac{8wg}{w} \times (\text{AD} - \text{AE})$. And hence $v = \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AE})]}$ the general expression for the velocity at E. And when E arrives at C, it gives the greatest velocity there $= \sqrt{[\frac{8wg}{w} \times (\text{AD} - \text{AC})]}$. Which, when $\text{W} = 28800$, $w = 1$, $2l = 3$ feet, and $\text{CD} = 6$ inches or $\frac{1}{2}$ a foot, is $\sqrt{(8 \times 28800 \times 16\frac{1}{12} \times \frac{\sqrt{10}-3}{2})} = 548\frac{1}{3}$ feet per second. Which came out $555\frac{7}{10}$ in the last problem, by using always AC for AE in the value of f. But when the extent of the vibrations is very small, as $\frac{1}{10}$ of an inch, as it commonly is, this greatest velocity here will be $\sqrt{8 \times 28800 \times 16\frac{1}{12} \times \frac{1}{43200}} = 9\frac{1}{4}$ nearly, which in the last problem was $9\frac{5}{19}$ nearly.

To find the time, it is $\dot{t}$ or $\frac{-\dot{x}}{v} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{[c - \sqrt{(l^2 + x^2)}]}}$, making $c = \text{AD} = \sqrt{(l^2 + a^2)}$. To find the fluent the easier, multiply the numer. and denom. both by $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$, so shall $\dot{t} = \sqrt{\frac{w}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}} \times \sqrt{[c + \sqrt{(l^2 + x^2)}]}$. Expand now the quantity $\sqrt{[c + \sqrt{(l^2 + x^2)}]}$ in a series, and put $d = c + l$, so shall $\dot{t} = \sqrt{\frac{wd}{8wg}} \times \frac{-\dot{x}}{\sqrt{(a^2 - x^2)}}(1 + \frac{x^2}{4dl} - \frac{2d + l}{32d^3l^3}x^4 + \frac{4d^3 + 2dl + l^3}{128d^5l^5}x^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^7l^7}x^8$ &c). Now the fluent of the first term $\frac{\dot{x}}{\sqrt{(a^2 - x^2)}}$ is $=$ the arc to sine $\frac{x}{a}$ and radius 1, which arc call A; and let P, Q, be the fluents of any other two successive terms, without the coefficients, the distance of Q from the first term A being n; then it is evident that $\dot{\text{Q}} = x^2\dot{\text{P}} = x^{2n}\dot{\text{A}}$, and $\dot{\text{P}} = x^{2n-2}\dot{\text{A}}$. Assume therefore $\text{Q} = b\text{P} - ex^{2n-1}\sqrt{(a^2 - x^2)}$; then is $\dot{\text{Q}}$ or $x^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ex^{2n-2}\dot{x}\sqrt{(a^2 - x^2)} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - \frac{(2n - 1)ea^2x^{2n-2}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{(2n-1)ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} + \frac{ex^{2n}\dot{x}}{\sqrt{(a^2 - x^2)}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + (2n - 1)ex^2\dot{\text{P}} + ex^2\dot{\text{P}} = b\dot{\text{P}} - (2n - 1)ea^2\dot{\text{P}} + 2nex^2\dot{\text{P}}$. Then, comparing the coefficients of the like terms, we find $1 = 2en$, and $b = (2n - 1)ea^2$; from which are obtained $e = \frac{1}{2n}$, and $b = \frac{2n-1}{2n}a^2$. Consequently $\text{Q} = \frac{(2n - 1)a^2\text{P} - x^{2n-1}\sqrt{(a^2 - x^2)}}{2n}$, the general

equation between any two successive terms, and by means of which the series may be continued as far as we please. And hence, neglecting the coefficients, putting A = the first term, namely, the arc whose sine is $\frac{x}{a}$, and B, C, D, &c, the following terms, the series is as follows, $A + \frac{a^2A - x\sqrt{(a^2 - x^2)}}{2} + \frac{3a^2B - x^3\sqrt{(a^2 - x^2)}}{4} + \frac{5a^2C - x^5\sqrt{(a^2 - x^2)}}{6}$ &c. Now when $x = 0$, this series $= 0$; and when $x = a$, the series becomes $\frac{1}{2}p + \frac{a^2A}{2} + \frac{3a^2B}{4} + \frac{5a^2C}{6}$ &c, where $p = 3{\cdot}1416$, or the series is $\frac{1}{2}p(1 + \frac{1}{2}a^2 + \frac{1 \cdot 3}{2 \cdot 4}a^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}a^6$ &c.)



So that, by taking in the coefficients, the general time of passing over any distance DE will be

$$\sqrt{\frac{w(c+l)}{8wg}} \times \tfrac{1}{2}p \times \left(1 + \frac{1}{4dl} \cdot \tfrac{1}{2}a^2 - \frac{2d + l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4}a^4 \text{ \&c}, - \text{arc sin. } \frac{x}{a} - \frac{1}{4dl} \cdot \frac{a^2A - x\sqrt{(a^2 - x^2)}}{2} + \frac{2d + l}{32d^2l^3} \cdot \frac{3a^2B - x^3\sqrt{(a^2 - x^2)}}{4} \text{ \&c.}\right.$$

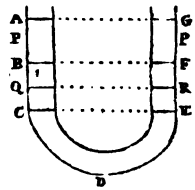
And hence, taking $x = 0$, and doubling, the time of a whole vibration, or double the time of passing over CD will be equal to $\frac{1}{2}p\sqrt{\frac{w(c+l)}{2wg}} \times (1 + \frac{1}{4dl} \cdot \frac{1}{2}a^2 - \frac{2d + l}{32d^2l^3} \cdot \frac{1 \cdot 3}{2 \cdot 4}a^4 + \frac{4d^3 + 2dl + l^3}{128d^3l^5} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}a^6 - \frac{40d^3 + 8d^2l + 12dl^2 + 5l^3}{2048d^4l^7} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}a^8$ &c.)

Which, when $a = 0$, or $c = l$, becomes only $\frac{1}{2}p\sqrt{\frac{wl}{wg}}$, the same as in the last problem, as it ought.

Taking here the same numbers as in the last problem, viz, $l = \frac{3}{4}$, $a = \frac{1}{2}$, $w = 2$, $W = 28800$, $g = 16\frac{1}{12}$; then $\frac{1}{2}p\sqrt{\frac{w(c + l)}{2wg}} = {\cdot}0040514$, and the series is $1 + {\cdot}003762 - {\cdot}000175 + {\cdot}000003$ &c $= 1{\cdot}006590$; therefore ${\cdot}0040514 \times 1{\cdot}006590 = {\cdot}0040965 = \frac{1}{245\frac{1}{5}}$ is the time of one whole vibration, and consequently $245\frac{1}{5}$ vibrations are performed in a second; which were 250 in the last problem.

PROBLEM XXIII.

It is proposed to determine the Velocity, and the Time of Vibration, of a Fluid in the Arms of a Canal or bent Tube.



Let the tube ABCDEF have its two branches AC, GE vertical, and the lower part CDE in any position whatever, the whole being of a uniform diameter or width throughout. Let water, or quicksilver, or any other fluid, be poured in, till it stand in equilibrio, at any horizontal line BF. Then let one surface be pressed or pushed down by shaking, from B to C, and the other will ascend through the equal space FG; after which let them be permitted freely to return. The surfaces will then continually vibrate in equal times between AC and EG. The velocity and times of which oscillations are therefore required.

When the surfaces are any where out of a horizontal line, as at P and Q, the parts of the fluid in QDR, on each side, below QR, will balance each other; and the weight of the part in PR, which is equal to 2PF, gives motion to the whole. So that the weight of the part 2PF is the motive force by which the whole fluid is urged, and therefore $\frac{\text{wt. of } 2PF}{\text{whole wt.}}$ is the accelerative force. Which weights being proportional to their lengths, if l be the length of the whole fluid, or axis of the tube filled, and $a =$ FG or BC; then is $\frac{a}{l}$ the accelerative force. Putting theref. $x =$ GP any variable distance, v the velocity, and t the time; then PF $= a - x$, and $\frac{2a-2x}{l} = f$ the accelerative force; hence vv or $2gfs = \frac{4g}{l}(a\dot{x} - x\dot{x})$; the fluents of which give $v^2 = \frac{4g}{l}(2ax - x^2)$, and $v = \sqrt{(4g \times \frac{2ax - x^2}{l})}$ is the general expression for the velocity at any term. And when $x = a$, it becomes $v = 2a\sqrt{\frac{g}{l}}$ for the greatest velocity at B and F.

Again, for the time, we have $\dot{t}$ or $\frac{\dot{s}}{v} = \frac{1}{2}\sqrt{\frac{l}{g}} \times \frac{\dot{x}}{\sqrt{(2ax - x^2)}}$; the fluents of which give $t = \frac{1}{2}\sqrt{\frac{l}{g}} \times$ arc to versed sine $\frac{x}{a}$ and radius 1, the general expression for the time. And when $x = a$, it becomes $t = \frac{1}{4}p\sqrt{\frac{l}{g}}$ for the time of moving from G to F, p being $= 3{\cdot}1416$; and consequently $\frac{1}{2}p\sqrt{\frac{l}{g}}$ the time of a whole vibration from G to E, or from C to A. And which therefore is the same, whatever AB is, the whole length l remaining the same.

And the time of vibration is also equal to the time of the vibration of a pendulum whose length is $\frac{1}{2}l$, or half the length of the axis of the fluid. So that, if the length l be $78\frac{1}{4}$ inches, it will oscillate in 1 second.

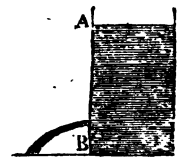
Scholium. This reciprocation of the water in the canal, is nearly similar to the motion of the waves of the sea. For the time of vibration is the same, however short the branches are, provided the whole length be the same. So that when the height is small, in proportion to the length of the canal, the motion is similar to that of a wave, from the top to the bottom or hollow, and from the bottom to the top of the next wave; being equal to two vibrations of the canal; the whole length of a wave, from top to top, being double the length of the canal. Hence the wave will move forward by a space nearly equal to its breadth, in the time of two vibrations of a pendulum whose length is ($\frac{1}{2}l$) half the length of the canal, or one-fourth of the breadth of a wave, or in the time of one vibration of a pendulum whose length is the whole breadth of the wave, since the times of vibration are as the square roots of their lengths. Consequently, waves whose breadth is equal to $39\frac{1}{8}$ inches, or $3\frac{25}{96}$ feet, will move over $3\frac{25}{96}$ feet in a second, or $195\frac{5}{8}$ feet in a minute, or nearly 2 miles and a quarter in an hour. And the velocity of greater or less waves will be increased or diminished in the subduplicate ratio of their breadths.

Thus, for instance, for a wave of 18 inches breadth, as $\sqrt{39\frac{1}{8}} : 39\frac{1}{8} :: \sqrt{18} : \sqrt{(39\frac{1}{8} \times 18)} = \frac{3}{2}\sqrt{313} = 26{\cdot}5377$ the velocity of the wave of 18 inches breadth.

PROBLEM XXIV.

To assign the Velocity with which Water, or other Fluids, spouts out from the bottom of a Vessel.

The velocity with which water runs out by a hole in the bottom or side of a vessel, is equal to that which is generated by gravity through the height of the fluid above the hole; that is, the velocity of a heavy body acquired by falling freely through the height AB.



For, divide the altitude AB into a great number of very small parts, each being 1, their number a, or a = the altitude AB.

Now, by prop. 61, vol. 2 of the Course, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

Because the momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the

same or an equal time, are equal. And as the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in 1 second, in the spouting water: also $2g$ or $32\frac{1}{5}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times 2g$, or $2ahg$, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = 2ahg$: hence then $v^2 = 4ag$, and $v = 2\sqrt{ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid.

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $2\sqrt{ag} = 2\sqrt{27720} \times 16\frac{1}{12} =$ 1335 feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

Corol. 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{\text{AB}}$.

Corol. 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the

friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

Corol. 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}h\sqrt{a}$ cubic feet, when a and h are taken in feet.

So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{ag} = 2\sqrt{25} \times 193 =$ 139 cubic inches, which is the quantity that would be discharged per second.

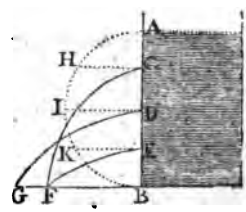
SCHOLIUM.

When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next problem.

It is not to be expected that experiments, on the quantity of water run out, will exactly agree with this theory, as well on account of the resistance of the air, as the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water on entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the sur-

rounding parts of the fluid, which is in commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifice pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments, on the quantity of water discharged through apertures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through half the height of the fluid above the orifice.

Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet spouts on the horizontal plane BG, are as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the ranges BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{(AC . CB)}$; and the distance BG as $\sqrt{(AD . DB)}$. And hence, if two holes be made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle



AC . CB is equal the rectangle AE . EB: which makes BF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows that the projection from the middle point D will be farthest, for DI is the greatest ordinate.

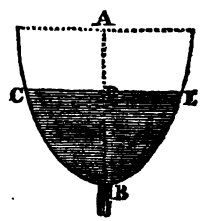
These are the proportions of the distances: but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{(AC . CB)} = 2CH =$ BF, the space ranged on the horizontal plane. And the greatest range BG = 2DI, or 2AD, or equal to AB.

And as these ranges answer very exactly to the experiments, this confirms the theory, as to the velocity assigned.

PROBLEM XXIV.

To assign the Time of emptying a Vessel of Water, or other Fluid, through a Hole in the Bottom, as ACBE.

Put h = AB the first height of the fluid above the hole;
x = DB the variable altitude at any time;
a = the area of the orifice B;
a' = the area of the descending surface CE.



Now the velocity of any issuing uniform fluid, being the same as that acquired by a body in falling through the height DB of the fluid, which is as the square-root of the height; and 32 being the velocity acquired in falling through the space

16 feet; therefore $\sqrt{16} : 32 :: \sqrt{DB}$ or $\sqrt{x} : 8\sqrt{x}$ the velocity of falling through DB, or of the issuing fluid at B; and $8a\sqrt{x}$ the quantity issuing in 1 second; or rather, reducing this by $\frac{3}{4}$, then $4 : 3 :: 8a\sqrt{x} : 6a\sqrt{x}$ for the quantity issuing per second, allowing $\frac{3}{4}$ for the contraction of the stream. But the content of the space descended by the surface CE, is always equal to what issues by the orifice at B; therefore the velocities at B and of the descending section CE, are always reciprocally proportional, that is, $a' : a :: 6\sqrt{x} : \frac{6a\sqrt{x}}{a'}$ the velocity, per second, of the descending section. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents, or uniformly described spaces, are as their times; therefore $\frac{6a\sqrt{x}}{a'} : \dot{x} :: 1'' : \frac{a'\dot{x}}{6a\sqrt{x}} = \dot{t}$, the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting.

As to the fluent of this fluxion, it will be various, according to the figure of the vessel. If this be a prism of any kind whatever, the quantity a', or the section CE, will be always the same constant quantity; and then the fluent will be $\frac{a'}{3a}\sqrt{x} = t$ the time of exhausting the prism, whose base is a', and altitude x.

But, for any other figure, the quantity a', or CE, will be a variable quantity, and will alter the form of the fluent, according to the nature of the figure. So, if the vessel be a paraboloid for example, the nature of which is such, that the different sections are proportional to their distances from the vertex of the curve; that is, as BA : b the section at A : : BD : CE, or as $h : b :: x : \frac{bx}{h} = a'$, the value of a'; which being substituted for it in the general value of $\dot{t}$, it becomes $\dot{t} = \frac{b\sqrt{x}}{6ah}\dot{x}$; the whole fluent of which is $\frac{b\sqrt{h}}{9a}$, for the time of the complete exhausting of the vessel ACBE at the vertex downwards.

But if the vertex were upwards, the fluxion of the time would be $\dot{t} = \frac{h-x}{\sqrt{x}} \times \frac{b\dot{x}}{6ah}$, the whole fluent of which would

be $t = \frac{2b\sqrt{h}}{9a}$, for the time of exhausting the paraboloid at the base, or the vertex upwards. Hence it may be observed, that this latter time is just double the former, when the vertex was downwards; also that this last, with the vertex downwards, is but the $\frac{1}{3}$ part of the time for the prism of equal base and altitude first above found; or that the three times are in proportion respectively as the three numbers 3, 2, 1, having all the same base and altitude; also that all their times are proportional to the base and root of the altitude directly, and as the orifice inversely.

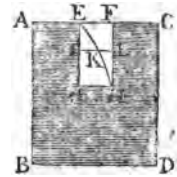
The times for many other figures may be seen in the first article in my Mathematical Miscellany.

In the above investigation, the resistance made by the air to the issuing fluid, is neglected, the issue being supposed to be made in a vacuum. But the difference of effect ought not to be neglected, when the density of the medium bears any considerable proportion to that of the issuing fluid. It will make no difference in the account, in whatever part of the base the aperture is placed, the altitude of the surface above it being the only consideration. Nor is it material what the figure of it is; whether circular, square, triangular, &c, or regular or not; its area alone being the only necessary consideration.

PROBLEM XXV.

To determine the Issue of a Fluid through a Notch at the Top of a Vessel.

If a notch or slit EH in form of a parallelogram, be cut in the side of a vessel, full of water, AD; the quantity of water flowing through it, will be $\frac{2}{3}$ of the quantity flowing through an equal orifice, placed at the whole depth EG, or at the base GH, in the same time; it being supposed that the vessel is always kept full.



For, the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

Corol. 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

Corol. 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

PROBLEM XXVI.

To determine the Time of filling the Ditches of a Work with Water at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{16} : \sqrt{10} :: 32 : 8\sqrt{10}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $32\sqrt{10}$ is the quantity per second running through it; or $24\sqrt{10}$ only when reduced $\frac{1}{4}$ for the contraction of the stream; consequently $24\sqrt{10} : 189000 :: 1'' : \frac{7875}{\sqrt{10}} = 2490''$ or 41′ 10″ nearly, is the time of filling the ditch.

PROBLEM XXVII.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom, the Height of the Aperture being very small in respect of the Altitude of the Fluid.

Put $a =$ the area of the aperture or sluice;
$d =$ the whole depth of water;
$x =$ the variable alt. of the surface above the aperture;
$a' =$ the area of the surface of the water.

Then $\sqrt{16} : \sqrt{x} :: 32 : 8\sqrt{x}$ the velocity with which the fluid will issue at the sluice; and hence $a' : a :: 8\sqrt{x} : \frac{8a\sqrt{x}}{a'}$ the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there, or it is only $\frac{6a\sqrt{x}}{a'}$ when reduced $\frac{1}{4}$ for the contraction of the stream. Now in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{6a\sqrt{x}}{a'} : \dot{x} :: 1'' : \frac{a'\dot{x}}{6a\sqrt{x}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-a'\dot{x}}{6a\sqrt{x}}$.

Now, when the nature or figure of the vessel is given, there will be given a' in terms of x; which value of a' being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

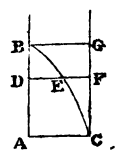
So if, for example, the vessel be any prism, or every where of the same breadth; then a' is a constant quantity, and therefore the fluent is $-\frac{a'}{3a}\sqrt{x}$. But when $x = d$, this becomes $-\frac{a'}{3a}\sqrt{d}$, and should be 0; therefore the correct fluent is $t = \frac{a'}{3a} \times (\sqrt{d} - \sqrt{x})$ for the time of the surface descending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{a'}{3a}\sqrt{d}$.

And hence if a' be 10000 square feet, $a = 1$ square foot, and $d = 10$ feet; the time is 10540 seconds, or $2^h\ 55'\ 40''$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90 + x}{90} \times 20$ the breadth of the surface of the water when its depth in the canal is x; and therefore $a = \frac{90 + x}{90} \times 20000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-a'\dot{x}}{6a\sqrt{x}} = 10000 \times \frac{90 + x}{90} \times \frac{-\dot{x}}{3a\sqrt{x}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $\frac{10000}{3} \times \frac{180 + \frac{2}{3}d}{90a} \times \sqrt{d} = \frac{10000 \times 186 \times 3}{90 \times 3} = 20666''$ nearly, or $5^h\ 44'\ 26''$, the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XXVIII.

To determine the Time of emptying any Ditch, or Inundation, &c, by a Cut or Notch, from the Top to the Bottom of it.



Let x = AB, the variable height of the descending water at any time;
b = AC, the breadth of the cut;
d = the whole or first depth of water;
A = the area of the surface of the water in the ditch;
$g = 16\frac{1}{12}$ feet, the descent by gravity in 1″.

Now, the velocity at any point D, is as $\sqrt{\text{BD}}$, that is as the ordinate DE of a parabola BEC, whose base is AC, and altitude AB. Therefore the velocities at all the points in AB, are as all the ordinates of the parabola. Consequently, the quantity of water running through the cut ABGC, in any time, is to the quantity which would run through an equal aperture placed all at the bottom, in the same time, as the area of the parabola ABC, to the area of the parallelogram ABGC, that is, as 2 to 3.

But $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$, the velocity at AC; therefore $2\sqrt{gx} \times bx \times \frac{2}{3} = \frac{4}{3}bx\sqrt{gx}$ is the quantity discharged per second through ABGC; and consequently $\frac{4bx\sqrt{gx}}{3\text{A}}$ is the velocity per second of the descending surface. Hence then

$\frac{4bz\sqrt{gx}}{3A} : -\dot{x} :: 1'' : \frac{-3A\dot{x}}{4bz\sqrt{gx}} = \dot{t}$, the fluxion of the time of descending.

Now when A the surface of the water is constant, or the ditch is equally broad throughout, the correct fluent of this fluxion gives $t = \frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d}-\sqrt{x}}{\sqrt{dx}}$ for the general time of sinking the surface to any depth x. And when $x = 0$, this expression is infinite; which shows that the time of a complete exhaustion is infinite.

But if $d = 9$ feet, $b = 2$ feet, $A = 21 \times 1000 = 21000$, and it be required to exhaust the water down to $\frac{1}{10}$ of a foot deep; then $x = \frac{1}{10}$, and the above expression becomes $\frac{3 \times 21000}{4 \times 4\frac{1}{8}} \times \frac{3-\frac{1}{4}}{\frac{3}{4}} = 14400''$, or just 4 hours for that time. And if it be required to depress it 8 feet, or till 1 foot depth of water remain in the ditch, the time of sinking the water to that point will be 43′ 38″.

Again, if the ditch be the same depth and length as before, but 20 feet broad at bottom, and 22 at top; then the descending surface will be a variable quantity, and, by prob. 27, it will be $\frac{90+x}{90} \times 20000$; hence in this case the flux. of the time, or $\frac{-3A\dot{x}}{4bz\sqrt{gx}}$, becomes $\frac{-500}{3b\sqrt{g}} \times \frac{90+x}{x\sqrt{x}}\dot{x}$; the correct fluent of which is $t = \frac{1000}{3b\sqrt{g}} \times \left(\frac{90-x}{\sqrt{x}} - \frac{90-d}{\sqrt{d}}\right)$ for the time of sinking the water to any depth x.

Now when $x = 0$, this expression for the complete exhaustion becomes infinite.

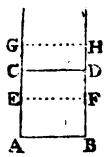
But if . . $x = 1$ foot, the time t is 42′ 56″$\frac{1}{2}$.

And when $x = \frac{1}{10}$ foot, the time is 3^h 50′ 28″$\frac{1}{2}$.

PROBLEM XXIX.

To determine the Time of filling the Ditches of a Fortification 6 Feet deep with Water, through the Sluice of a Trunk of 3 Feet Square, the Bottom of which is level with the Bottom of the Ditch, and the Height of the supplying Water is 9 Feet above the Bottom of the Ditch.

Let ACDB represent the area of the vertical sluice, being a square of 9 square feet, and AB level with the bottom of the ditch. And suppose the ditch filled to any height AE, the surface being then at EF.



Put $a = 9$ the height of the head or supply;
$b = 3 =$ AB $=$ AC;
$g = 16\frac{1}{12}$;
A $=$ the area of a horizontal section of the ditches;
$x = a -$ AE, the height of the head above EF.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the water presses through the part AEFB; and theref. $2\sqrt{gx} \times$ AEFB $= 2b\sqrt{gx}\,(a - x)$ is the quantity per second running through AEFB. Also, the quantity running per second through ECDF is $2\sqrt{gx} \times \frac{11}{12}$ECDF $= \frac{11}{6}b\sqrt{gx}(b - a + x)$ nearly. For the real quantity is, by proceeding as in the last prob. the difference between two parab. segs. the alt. of the one being x, its base b, and the alt. of the other $a - b$; and the medium of that dif. between its greatest state at AB, where it is $\frac{9}{10}$AD, and its least state at CD, where it is 0, is nearly $\frac{11}{12}$ED. Consequently the sum of the two, or $\frac{1}{6}b\sqrt{gx}$ $(a + 11b - x)$ is the quantity per second running in by the whole sluice ACDB. Hence then $\frac{1}{6}b\sqrt{gx} \times \frac{a + 11b - x}{A} = v$ is the rate or velocity per second with which the water rises in the ditches; and so $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-6A}{b\sqrt{g}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{c - x}$ the fluxion of the time of filling to any height AE, putting $c = a + 11b$.

Now when the ditches are of equal width throughout, A is a constant quantity, and in that case a correct fluent of this fluxion is $t = \frac{6A}{b\sqrt{gc}} \times \log.\left(\frac{\sqrt{c} + \sqrt{a}}{\sqrt{c} - \sqrt{a}} \times \frac{\sqrt{c} - \sqrt{x}}{\sqrt{c} + \sqrt{x}}\right.$ the general expression for the time of filling to any height AE, or $a - x$, not exceeding the height AC of the sluice. And when $x =$ AC $= a - b = d$ suppose, then $t = \frac{6A}{b\sqrt{gc}} \times \log.$

$(\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})$ is the time of filling to CD the top of the sluice.

Again, for filling to any height GH above the sluice, x denoting as before $a - $ AG the height of the head above GH, $2\sqrt{gx}$ will be the velocity of the water through the whole sluice AD: and therefore $2b^2\sqrt{gx}$ the quantity per second, and $\frac{2b^2\sqrt{gx}}{A} = v$ the rise per second of the water in the ditches; consequently $v : -\dot{x} :: 1'' : \dot{t} = -\frac{\dot{x}}{v} = \frac{-A}{2b^2\sqrt{g}} \times \frac{\dot{x}}{\sqrt{x}}$ the general fluxion of the time; the correct fluent of which, being 0 when $x = a - b = d$, is $t = \frac{A}{b^2\sqrt{g}}(\sqrt{d} - \sqrt{x})$ the time of filling from CD to GH.

Then the sum of the two times, namely, that of filling from AB to CD, and that of filling from CD to GH, is $\frac{A}{b\sqrt{g}}[\frac{\sqrt{d}-\sqrt{x}}{b} + \frac{6}{\sqrt{x}} \log. (\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}+\sqrt{d}})]$ for the whole time required. And, using the numbers in the prob., this becomes $\frac{A}{3\sqrt{g}}[\frac{\sqrt{6}-\sqrt{3}}{3} + \frac{6}{\sqrt{42}} \times 1.(\frac{\sqrt{42}+\sqrt{9}}{\sqrt{42}-\sqrt{9}} \cdot \frac{\sqrt{42}-\sqrt{6}}{\sqrt{42}+\sqrt{6}})] =$ 0·03577277A, the time in terms of A the area of the length and breadth, or horizontal section of the ditches. And if we suppose that area to be 200000 square feet, the time required will be 7154″, or 1^h 59′ 14″.

And if the sides of the ditch slope a little, so as to be a little narrower at the bottom than at top, the process will be nearly the same, substituting for A its variable value, as in the preceding problem. And the time of filling will be very nearly the same as that above determined.

PROBLEM XXX.

But if the Water, from which the Ditches are to be filled, be the Tide, which at Low Water is below the Bottom of the Trunk, and rises to suppose 9 Feet above the Bottom of it by a regular Rise of One Foot in Half an Hour; it is required to

ascertain the Time of Filling it to 6 Feet high, as before in the last Problem.

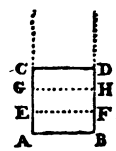
Let ACDB represent the sluice; and when the tide has risen to any height GH, below CD the top of the sluice, without the ditches, let FF be the mean height of the water within. And put $b = 3 =$ AB $=$ AC;

$g = 16\frac{1}{12}$;

A $=$ horizontal section of the ditches;

$x =$ AG;

$z =$ AE.



Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)}$ the velocity of the water through AEFB; and

$\sqrt{g} : \sqrt{\text{EG}} :: \frac{4}{3}g : \frac{4}{3}\sqrt{g(x-z)}$ the mean vel. through EGHF; theref. $2bz\sqrt{g(x-z)}$ is the quantity per sec. through AEFB; and $\frac{4}{3}b(x-z)\sqrt{g(x-z)}$ is the same through EGHF; conseq. $\frac{2}{3}b\sqrt{g} \times (2x+z)\sqrt{(x-z)}$ is the whole through AGHB per second. This quantity divided by the surface A, gives $\frac{2b\sqrt{g}}{3\text{A}} \times (2x+z)\sqrt{(x-z)} = v$ the velocity per second with which EF, or the surface of the water in the ditches, rises. Therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}.$$

But as GH rises uniformly 1 foot in 30′ or 1800″, therefore $1 :$ AG $:: 1800'' : 1800x = t$ the time of the tide rising through AG; conseq. $\dot{t} = 1800\dot{x} = \frac{3\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(2x+z)\sqrt{(x-z)}}$, or $m\dot{z} = (2x+z)\sqrt{(x-z)}\,.\,\dot{x}$ is the fluxional equa. expressing the relation between x and z; where $m = \frac{\text{A}}{1200b\sqrt{g}} = \frac{3200}{231}$ or $13\frac{197}{231}$ when A $= 200000$ square feet.

Now to find the fluent of this equation, assume $z = \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{9}{2}} + \text{C}x^{\frac{13}{2}} + \text{D}x^{\frac{17}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{4}{2}} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{7}{2}} - \frac{\text{A}^3+4\text{AB}+8\text{C}}{16}x^{\frac{10}{2}} \text{ \&c,}$$

$$2x + z = 2x + \text{A}x^{\frac{5}{2}} + \text{B}x^{\frac{9}{2}} + \text{C}x^{\frac{13}{2}} \text{ \&c,}$$

$$(2x+z)\sqrt{(x-z)}\dot{x} = 2x^{\frac{3}{2}}\dot{x} - \frac{3\text{A}^2}{4}x^{\frac{9}{2}}\dot{x} - \frac{\text{A}^3+6\text{AB}}{4}x^{\frac{12}{2}}\dot{x} \text{ \&c,}$$

and $m\dot{z} = \frac{5}{2}m\text{A}x^{\frac{3}{2}}\dot{x} + \frac{9}{2}m\text{B}x^{\frac{6}{2}}\dot{x} + \frac{11}{2}m\text{C}x^{\frac{9}{2}}\dot{x} + \frac{14}{2}m\text{D}x^{\frac{12}{2}}\dot{x}$ &c.

Then equating the coefficients of the like terms,

so shall	and consequently
$\frac{5}{2}m\text{A} = 2,$	$\text{A} = \frac{4}{5m},$
$\frac{9}{2}m\text{B} = 0,$	$\text{B} = 0,$
$\frac{11}{2}m\text{C} = -\frac{3}{4}\text{A}^2,$	$\text{C} = -\frac{24}{275m^3},$
$\frac{14}{2}m\text{D} = -\frac{3}{4}\text{A}^3 - \frac{3}{2}\text{AB},$	$\text{D} = -\frac{16}{875m^4},$
&c;	&c.

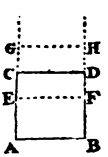
Which values of A, B, C, &c, substituted in the assumed value of z, give

$$z = \frac{4}{5m}x^{\frac{5}{2}} - \frac{24}{275m^3}x^{\frac{11}{2}} - \frac{16}{875m^4}x^{\frac{14}{2}} \text{ \&c};$$

$$\text{or } z = \frac{4}{5m}x^{\frac{5}{2}} \text{ very nearly.}$$

And when $x = 3 =$ AC, then $z = \cdot886$ of a foot, or $10\frac{2}{3}$ inches, $=$ AE, the height of the water in the ditches when the tide is at CD or 3 feet high without, or in the first hour and half of time.

Again, to find the time, after the above, when EF arrives at CD, or when the water in the ditches arrives as high as the top of the sluice.



The notation remaining as before,

then $2bz\sqrt{g(x-z)}$ per sec. runs through AF,

and $\frac{8}{3}b(3-z)\sqrt{g(x-z)}$ per sec. thro' ED nearly;

therefore $\frac{2}{3}b\sqrt{g}\times(12+z)\sqrt{(x-z)}$ is the whole per second through AD nearly.

conseq. $\frac{2b\sqrt{g}}{5\text{A}} \times (12+z)\sqrt{(x-z)} = v$ is the velocity per second of the point E; and therefore

$$v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{5\text{A}}{2b\sqrt{g}} \times \frac{\dot{z}}{(12+z)\sqrt{(x-z)}} = 1800\dot{x}, \text{ or}$$

$$m\dot{z} = (12+z)\sqrt{(x-z)}\,.\,\dot{x}, \text{ where } m = \frac{\text{A}}{720b\sqrt{g}} = 23\tfrac{2}{23} \text{ nearly.}$$

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c. So shall

$$\sqrt{(x-z)} = x^{\frac{1}{2}} - \frac{\text{A}}{2}x^{\frac{2}{2}} - \frac{\text{A}^2+4\text{B}}{8}x^{\frac{3}{2}} - \frac{\text{A}^3+4\text{AB}+8\text{C}}{16}x^{\frac{4}{2}} \text{ \&c};$$

$$12 + z = 12 + \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} \text{ \&c};$$

$(12+z).\sqrt{(x-z)}.\dot{x} = 12x^{\frac{1}{2}}\dot{x} - 6\text{A}x^{\frac{2}{2}}\dot{x} - (\tfrac{3}{2}\text{A}^2 + 6\text{B})x^{\frac{3}{2}}\dot{x}$ &c;

$m\dot{z} = \tfrac{3}{2}m\text{A}x^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}m\text{B}x^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}m\text{C}x^{\frac{3}{2}}\dot{x}$ &c.

Then, equating the like terms, &c, we have

$\text{A} = \frac{8}{m}$, $\text{B} = -\frac{24}{m^2}$, $\text{C} = \frac{96}{5m^3}$, $\text{D} = \frac{64}{3m^4}$ nearly, &c.

Hence $z = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 + \frac{96}{5m^3}x^{\frac{5}{2}} + \frac{64}{3m^4}x^3$ &c.

Or $z = \frac{8}{m}x^{\frac{3}{2}}$ nearly.

But, by the first process, when $x = 3$, $z = \cdot 886$; which substituted for them, we have $z = \cdot 886$, and the series $= 1\cdot 63$; therefore the correct fluents are

$$z - \cdot 886 = -1\cdot 63 + \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c,}$$

$$\text{or } z + \cdot 744 = \frac{8}{m}x^{\frac{3}{2}} - \frac{24}{m^2}x^2 \text{ \&c.}$$

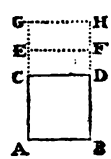
And when $z = 3 = \text{AC}$, it gives $x = 6\cdot 369$ for the height of the tide without, when the ditches are filled to the top of the sluice, or 3 feet high; which answers to $3^{\text{h}}\ 11'\ 4''$.

Lastly, to find the time of rising the remaining 3 feet above the top of the sluice; let

$x =$ CG the height of the tide above CD,

$z =$ CE ditto in the ditches above CD;

and the other dimensions as before.



Then $\sqrt{g} : \sqrt{\text{EG}} :: 2g : 2\sqrt{g(x-z)} =$ the velocity with which the water runs through the whole sluice AD; conseq. $\text{AD} \times 2\sqrt{g(x-z)} = 18\sqrt{g(x-z)}$ is the quantity per second running through the sluice, and $\frac{18\sqrt{g}}{\text{A}}\sqrt{(x-z)} = v$ the velocity of z, or the rise of the water in the ditches, per second; hence $v : \dot{z} :: 1'' : \dot{t} = \frac{\dot{z}}{v} = \frac{\text{A}}{18\sqrt{g}} \times \frac{\dot{z}}{\sqrt{(x-z)}} = 1800\dot{x}$, and $m\dot{z} = \dot{x}\sqrt{(x-z)}$ is the fluxional equation; where $m = \frac{\text{A}}{180^2\sqrt{g}} = \frac{3200}{2079}$.

To find the fluent,

Assume $z = \text{A}x^{\frac{3}{2}} + \text{B}x^{\frac{4}{2}} + \text{C}x^{\frac{5}{2}} + \text{D}x^{\frac{6}{2}}$ &c.

Then $x - z = x - \text{A}x^{\frac{3}{2}} - \text{B}x^{\frac{4}{2}} - \text{C}x^{\frac{5}{2}}$ &c,

$$\dot{x}\sqrt{(x-z)} = x^{\frac{1}{2}}\dot{x} - \frac{A}{2}x^{\frac{2}{2}}\dot{x} - \frac{A^2+4B}{8}x^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

$$m\dot{z} = \tfrac{1}{2}nAx^{\frac{1}{2}}\dot{x} + \tfrac{4}{2}nBx^{\frac{2}{2}}\dot{x} + \tfrac{5}{2}nCx^{\frac{3}{2}}\dot{x} \text{ \&c.}$$

Then equating the like terms gives

$$A = \frac{2}{3n},\ B = \frac{-1}{6n^2},\ C = \frac{1}{90n^3},\ D = \frac{-1}{810n^4}, \text{ \&c.}$$

Hence $z = \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2 + \frac{1}{90n^3}x^{\frac{5}{2}} - \frac{1}{810n^4}x^3$ &c.

But by the second case, when $z = 0$, $x = 3{\cdot}369$, which being used in the series, it is $1{\cdot}936$; therefore the correct fluent is $z = -1{\cdot}936 + \frac{2}{3n}x^{\frac{3}{2}} - \frac{1}{6n^2}x^2$ &c. And when $z = 3$, $x = 7$; the heights above the top of the sluice, answering to 6 and 10 feet above the bottom of the ditches. That is, for the water to rise to the height of 6 feet within the ditches, it is necessary for the tide to rise to 10 feet without, which just answers to 5 hours; and so long it would take to fill the ditches 6 feet deep with water, their horizontal area being 200000 square feet.

Further, when $x = 6$, then $z = 2{\cdot}117$ the height above the top of the sluice; to which add 3, the height of the sluice, and the sum $5{\cdot}117$, is the depth of water in the ditches in 4 hours and a half, or when the tide has risen to the height of 9 feet without the ditches.

Note. In the foregoing problems, concerning the efflux of water, it is taken for granted that the velocity is the same as that which is due to the whole height of the surface of the supplying water: a supposition which agrees with the principles of the greater number of authors: though some make the velocity to be that which is due to the half height only: and others make it still less.

Also in some places, where the difference between two parabolic segments was to be taken, in estimating the mean velocity of the water through a variable orifice, I have used a near mean value of the expression; which makes the operation of finding the fluents much more easy, and is at the same time sufficiently exact for the purpose in hand.

We may further add a remark here concerning the method of finding the fluents of the three fluxional forms that occur in the solution of this problem, viz, the three forms $m\dot{z} = (2x + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = (12 + z)\sqrt{(x - z)}\dot{x}$, and $m\dot{z} = \sqrt{(x - z)}\dot{x}$, the fluents of which are found by assuming the fluent mz in an infinite series ascending in terms of x with indeterminate coefficients A, B, C, &c, which coefficients are afterwards determined in the usual way, by equating the corresponding terms of two similar and equal series, the one series denoting one side of the fluxional equation, and the other series the other side. By similar series, is meant such as have equal or like exponents; though it is not necessary that the exponents of all the terms should be like or pairs, but only some of them, as those that are not in pairs will be cancelled or expelled by making their coefficients $= 0$ or nothing. Now the general way to make the two series similar, is to assume the fluent z equal to a series in terms of x, either ascending or descending, as here

$$z = x^r + x^{r+s} + x^{r+2s} \text{ \&c for ascending,}$$

$$\text{or } z = x^r + x^{r-s} + x^{r-2s} \text{ \&c for a descending}$$

series, having the exponents r, $r \pm s$, $r \pm 2s$, &c, in arithmetical progression, the first term r, and common difference s; without the general coefficients A, B, C, &c, till the values of the exponents be determined. In terms of this assumed series for z, find the values of the two sides of the given fluxional equation, by substituting in it the said series instead of z; then put the exponent of the first term of the one side equal that of the other, which will give the value of the first exponent r; in like manner put the exponents of the two 2d terms equal, which will give the value of the common difference s; and hence the whole series of exponents r, $r \pm s$, $r \pm 2s$, &c, becomes known.

Thus, for the last of the three fluxional equations above mentioned, viz, $m\dot{z} = \sqrt{(x-z)}\dot{x}$, or only $\dot{z} = \sqrt{(x - z)}\dot{x}$; having assumed as above $z = x^r + x^{r+s}$ &c, and taking the fluxion, then $\dot{z} = x^{r-1}\dot{x} + x^{r+s-1}\dot{x}$ + &c, omitting the coefficients; and the other side of the equation $\sqrt{(x - z)}\dot{x} =$

$\sqrt{(x-x^r-x^{r+s}\ \&c)} = x^{\frac{1}{2}}\dot{x} - x^{r-\frac{1}{2}}\dot{x}$ &c. Now the exponents of the first terms made equal, give $r-1=\frac{1}{2}$, theref. $r=1+\frac{1}{2}=\frac{3}{2}$; and those of the 2d terms made equal, give $r+s-1=r-\frac{1}{2}$, theref. $s-1=-\frac{1}{2}$, and $s=1-\frac{1}{2}=\frac{1}{2}$; conseq. the whole assumed series of exponents r, $r+s$, $r+2s$, &c, become $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, as assumed above in pa. 368.

Again, for the 2d equation $m\dot{x}$ or $\dot{z}=(12+z)\sqrt{(x-z)}\dot{x}$ $=(a+z)\sqrt{(x-z)}\dot{x}$; assuming $z=x^r+x^{r+s}$ &c as before, then $\dot{z}=x^{r-1}\dot{x}+x^{r+s-1}\dot{x}$ &c, and $\sqrt{(x-z)}\dot{x}=x^{\frac{1}{2}}\dot{x}-x^{r-\frac{1}{2}}\dot{x}$ &c, both as above; this mult. by $a+z$ or $a+x^r+x^{r+s}$ &c, gives $ax^{\frac{1}{2}}\dot{x}-ax^{r-\frac{1}{2}}\dot{x}$ &c: then equating the first exponents gives $r-1=\frac{1}{2}$ or $r=\frac{3}{2}$, and $r+s-1=r-\frac{1}{2}$ or $s=1-\frac{1}{2}=\frac{1}{2}$; hence the series of exponents is $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c, the same as the former, and as assumed in pa. 366.

Lastly, assuming the same form of series for z and $\dot{z}$ as in the above two cases, for the 1st fluxional equation also, viz, $m\dot{x}=(2x+z)\sqrt{(x-z)}\dot{x}$: then $\sqrt{(x-z)}\dot{x}=x^{\frac{1}{2}}\dot{x}-x^{r-\frac{1}{2}}\dot{x}$ &c, which mult. by $2x+z$, gives $2x^{\frac{3}{2}}\dot{x}-x^{r+\frac{1}{2}}\dot{x}$ &c: here equating the first exponents gives $r-1=\frac{3}{2}$, or $r=\frac{5}{2}$; and equating the 2d exponents gives $r+s-1=r+\frac{1}{2}$, or $s=\frac{3}{2}$; hence the series of exponents in this case is $\frac{5}{2}$, $\frac{8}{2}$, $\frac{11}{2}$, &c, as used for this case in pa. 365. Then, in every case, the general coefficients A, B, C, &c, are joined to the assumed terms x^r, x^{r+s}, &c, and the whole process conducted as in the three pages just referred to.

Such then is the regular and legitimate way of proceeding, to obtain the form of the series with respect to the exponents of the terms. But, in many cases we may perceive at sight, without that formal process, what the law of the exponents will be, as I indeed did in the solutions in the pages above referred to; and any person with a little practice may easily do the same.

PROBLEM XXXI.

To determine the Fall of the Water under the Arches of a Bridge.

The effects of obstacles placed in a current of water, such as the piers of a bridge, are, a sudden steep descent, and an increase of velocity in the stream of water, just under the arches, more or less in proportion to the quantity of the obstruction and velocity of the current: being very small and hardly perceptible where the arches are large and the piers few or small, but in a high and extraordinary degree at London-bridge, and some others, where the piers and the sterlings are so very large, in proportion to the arches. This is the case, not only in such streams as run always the same way, but in tide rivers also, both upward and downward, but much less in the former than in the latter. During the time of flood, when the tide is flowing upward, the rise of the water is against the under side of the piers; but the difference between the two sides gradually diminishes as the tide flows less rapidly towards the concluson of the flood. When this has attained its full height, and there is no longer any current, but a stillness prevails in the water for a short time, the surface assumes an equal level, both above and below bridge. But, as soon as the tide begins to ebb or return again, the resistance of the piers against the stream, and the contraction of the waterway, cause a rise of the surface above and under the arches, with a fall and a more rapid descent in the contracted stream just below. The quantity of this rise, and of the consequent velocity below, keep both gradually increasing, as the tide continues ebbing, till at quite low water, when the stream or natural current being the quickest, the fall under the arches is the greatest. And it is the quantity of this fall which it is the object of this problem to determine.

Now, the motion of free running water is the consequence

of, and produced by the force of gravity, as well as that of any other falling body. Hence the height due to the velocity, that is, the height to be freely fallen by any body to acquire the observed velocity of the natural stream, in the river a little way above bridge, becomes known. From the same velocity also will be found that of the increased current in the narrowed way of the arches, by taking it in the reciprocal proportion of the breadth of the river above, to the contracted way in the arches; viz, by saying, as the latter is to the former, so is the first velocity, or slower motion, to the quicker. Next, from this last velocity, will be found the height due to it as before, that is, the height to be freely fallen through by gravity, to produce it. Then the difference of these two heights, thus freely fallen by gravity, to produce the two velocities, is the required quantity of the waterfall in the arches; allowing however, in the calculation, for the contraction, in the narrowed passage, at the rate as observed by Sir I. Newton, in prop. 36 of the 2d book of the Principia, or by other authors, being nearly in the ratio of 25 to 21, or sufficiently near as 24 to 20, that is, as 6 to 5.

Such then are the elements and principles, on which the solution of the problem is easily made out as follows.

Let b = the breadth of the channel in feet;
v = mean velocity of the water in feet per second;
c = breadth of the waterway between the obstacles.

Now $6 : 5 :: c : \frac{5}{6}c$, the waterway contracted as above.

And $\frac{5}{6}c : b :: v : \frac{6b}{5c}v$, the velocity in the contracted way.

Also $32^2 : v^2 :: 16 : \frac{1}{64}v^2$, height fallen to gain the velocity v.

And $32^2 : (\frac{6b}{5c}v)^2 :: 16 : (\frac{6b}{5c})^2 \times \frac{1}{64}v^2$, ditto for the vel. $\frac{6b}{5c}v$.

Then $(\frac{6b}{5c})^2 \times \frac{v^2}{64} - \frac{v^2}{64}$ is the measure of the fall required.

Or $[(\frac{6b}{5c})^2 - 1] \times \frac{vv}{64}$ is a rule for computing the fall.

Or rather $\frac{1 \cdot 44b^2 - c^2}{64c^2} \times v^2$ very nearly, for the fall.

EXAM. 1. *For London-bridge.*

By the observations made by Mr. Labelye in 1746, The breadth of the Thames at London-bridge is 926 feet; The sum of the waterways at the time of low-water is 236 ft.; Mean velocity of the stream just above bridge is $3\frac{1}{6}$ ft. per sec. But under almost all the arches are driven into the bed great numbers of what are called dripshot piles, to prevent the bed from being washed away by the fall. These dripshot piles still further contract the waterways, at least $\frac{1}{6}$ of their measured breadth, or near 39 feet in the whole; so that the waterway will be reduced to 197 feet, or in round numbers suppose 200 feet.

Then $b = 926$, $c = 200$, $v = 3\frac{1}{6} = \frac{19}{6}$.

Hence $\frac{1\cdot44b^2 - c^2}{64c^2} = \frac{1234765 - 40000}{64 \times 40000} = \cdot467$.

And $v^2 = \frac{19^2}{6^2} = 10\frac{1}{36}$.

Theref. $\cdot467 \times 10\frac{1}{36} = 4\cdot63$ ft. $= 4$ ft. $8\frac{1}{2}$ in. the fall required. By the most exact observations made about the year 1736, the measure of the fall was 4 feet 9 inches.

EXAM. 2. *For Westminster-bridge.*

Though the breadth of the river at Westminster-bridge is 1220 feet; yet, at the time of the greatest fall, there is water through only the 13 large arches, which amount to but 820 feet; to which adding the breadth of the 12 intermediate piers, equal to 174 feet, gives 994 for the breadth of the river at that time; and the velocity of the water a little above the bridge, from many experiments, is not more $2\frac{1}{4}$ feet per second.

Here then $b = 994$, $c = 820$, $v = 2\frac{1}{4} = \frac{9}{4}$.

Hence $\frac{1\cdot44b^2 - c^2}{64c^2} = \frac{1422771 - 672400}{64 \times 672400} = \cdot017436$:

And $v^2 = \frac{81^2}{16} = 5\frac{1}{16}$.

Theref. $\cdot017436 \times 5\frac{1}{16} = \cdot08827$ ft. $= 1$ in. the fall required;

which is about half an inch more than the greatest fall observed by Mr. Labelye.

And, for Blackfriar's-bridge, the fall will be much the same as that of Westminster.

*** See other solutions at pa. 87, &c, vol. 1.

PROBLEM XXXII.

To determine the Circumstances relating to the Pressure of the Atmosphere on a given Space on the Earth, with the Height of a uniform Atmosphere, &c.

It is a fact, and may easily be shown, that the height is a constant quantity, or always the same, of a uniform atmosphere above any place, which shall be all of the uniform density with the air there, at any time, and whatever the weight of it then be as measured by the barometer. This property happens from the circumstance of the density of the air, at the earth's surface, always varying in proportion as indicated by the barometer; in fact, the height of the barometer at once shows both the density of the air and its weight, at the time. So that, as the density varies in exact proportion to the weight of the column, it therefore requires a column of the same height, in all cases, to make the respective weights or pressures; when the temperature or heat of the air is the same. Thus, if W and w denote the weights of atmosphere above any places, D and d their respective densities, and H, h the heights of the uniform columns, of these densities and weights: Then, each density multiplied by the height being equal to the weight, viz. $\text{H} \times \text{D} = \text{W}$, and $h \times d = w$; hence $\frac{\text{W}}{\text{D}} = \text{H}$, and $\frac{w}{d} = h$; but, as the weight is always proportional to the density, therefore $\frac{\text{W}}{\text{D}} = \frac{w}{d}$, that is, $\text{H} = h$, or the heights are equal.

The general height of the uniform atmospheric column is thus easily found. The weight of a cubic foot of mercury is 13600 ounces, or the pressure of a column of mercury 1

foot in height on a square foot of base; but the medium pressure of the atmosphere is equal to the column of 29·75 inches of the barometer; therefore 12 : 29·75 : : 13600 : 33717 ounces pressure of the atmosphere on a square foot; hence 33717 ÷ 144 = 234 ounces or $14\frac{5}{8}$ lb. is the medium pressure of the atmosphere on a square inch.

Again, for the height of a uniform atmosphere. Since 33717 ounces is the pressure of the atmosphere on a square foot; and $1\frac{2}{9}$ ounces is the weight of a cubic foot of air, or its pressure on a square foot of 1 foot in height; therefore $1\frac{2}{9}$: 1 : : 33717 : 27600 feet the height the atmosphere would be if it were all of the same uniform density as at the earth, being equal to $5\frac{1}{4}$ miles in height very nearly.

PROBLEM XXXIII.

To show that the Density of the Atmosphere, at different Heights above the Earth, Decreases in such Sort, that when the Heights Increase in Arithmetical Progression, the Densities Decrease in Geometrical Progression.

Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c, will be in geometrical progression decreasing.

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the

whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the series being that of n to m.

SCHOLIUM.

Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d, the dif. of alt. $A-a$ will be as the log. D − log. d. or log. $\frac{D}{d}$.

And if $A=0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer. For, by taking, with this instrument, the

pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so as that $a = h \times \log.\frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log.\frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$ $= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log.\frac{D}{d}$, or $= 63551 \times \log.\frac{M}{m}$ feet, or $10592 \times \log.\frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.—Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{120}$ part of an inch for every degree of heat, or rather $\frac{1}{3600}$ of the whole volume.

But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°. Thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the correspondent change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

PROBLEM XXXIV.

To divide a Given Circle into any proposed number of Equal Parts, by means of other Circles Concentric with the Given one.

This problem is now added here in the appendix, having been omitted in its proper place, Tract xiv. vol. 1, beside another problem, allied to this, as well in their nature as in their fate and consequences.

A particular case of the present problem was first of all, as far as I know, proposed in that useful and valuable little annual work, the Ladies' Diary, for the year 1709, in this form, viz, Seven men bought a grinding-stone, of 5 feet or 60 inches in diameter: and they agreed together, that each should grind off an equal share; so that one, beginning first, should grind his 7th part off the stone; then a second should

grind *his;* and so continue in succession. The question then was, how much of the diameter must each person grind down. In the year following an answer was so far given only, as merely to specify the numbers denoting the parts of the diameter to be ground down, or cut off, without any mode of solution whatever, either arithmetical or geometrical.

Many years after, a geometrical construction was given by a Mr. Hawney, in his little book on Mensuration, but so clumsy in its manner, as to require the description of a separate circle to ascertain the point through which each of the dividing concentric circles was to pass. And in this state it remained till about the year 1770, when Mr. James Ferguson, the ingenious lecturer on astronomy and mechanics, in his peregrinations came to Newcastle, where I then resided, to give the usual course of his public lectures; on which occasion, with the assistance of my friends, I not only procured him a numerous and respectable audience, but also accommodated him with the free use of the new school-rooms, which I had lately built, to deliver his lectures in. As Mr. F. commonly amused my family and friends at evenings, with showing his ingenious mechanical contrivances and drawings, on one of these occasions he produced a very neat and correct drawing, on a large scale, being a construction of this problem, in the very prolix way as before given by Hawney, but which he exhibited as a great curiosity. I ventured to remark to him that I thought a much simpler construction might be found out, for this problem, which was then new to me. As Mr. F. expressed a wish to see such a thing as a simpler construction, which however he seemed to have his doubts of procuring, I was induced to consider it that evening, before going to rest, and discovered the construction as follows.

The next morning I showed him the new and very simple construction, with its demonstration, which he seemed much pleased with, on account of the apparent simplicity, but doubted very much that it might not be correctly true.

On referring him to the accompanying demonstration, to satisfy himself of its geometrical truth, I was much surprized by his reply, that he could not understand that, but he would make the drawing correctly on a large scale, which was always his way to try if such things were true. In my surprize I asked where he had learned geometry, and by what Euclid or other book; to which he frankly replied he had never learned any geometry, nor could ever understand the demonstration of any one of Euclid's propositions. Accordingly the next morning, with a joyful countenance, he brought me the construction, neatly drawn out on a large sheet of pasteboard, saying he esteemed it a treasure, having found it quite right, as every point and line agreed to a hair's breadth, by measurement on the scale. This problem and the construction he afterwards inserted, with the proper acknowledgment, as a curiosity, in his Select Mechanical Exercises, p. 123, printed in 1773.

Also, in the beginning of the year 1771, when I commenced the republication of the Ladies' Diary Questions, I inserted the following solution of the question, at pa. 53 of the first volume of that work, viz.

"This question is to divide a circle, of 60 inches diameter, into 7 equal parts, or rings, bounded by concentric circles; of which the solution will be thus.—The whole circle, and each inner circle, after the several preceding rings are ground off, must be to each other, by the question, as the numbers 7, 6, 5, 4, 3, 2, 1; but circles are as the squares of their diameters; therefore the diameters of those circles will be to one another as $\sqrt{7}$, $\sqrt{6}$, $\sqrt{5}$, $\sqrt{4}$, $\sqrt{3}$, $\sqrt{2}$, $\sqrt{1}$: but the greatest diameter is 60 or $60\sqrt{\frac{7}{7}}$; therefore, by proportioning, all the other diameters will come out thus, $60\sqrt{\frac{6}{7}}$, $60\sqrt{\frac{5}{7}}$, $60\sqrt{\frac{4}{7}}$, $60\sqrt{\frac{3}{7}}$, $60\sqrt{\frac{2}{7}}$, $60\sqrt{\frac{1}{7}}$: Now the last of these is the diameter of the last person's share, and the difference between every two adjacent terms being taken, will give the double breadth of the rings, or the parts of the whole diameter to be ground off by the other persons; viz.

$60\sqrt{\tfrac{7}{7}}-60\sqrt{\tfrac{6}{7}}=\frac{\sqrt{7}-\sqrt{6}}{\sqrt{7}}\times 60=4{\cdot}450794$ the 1st person's share

$60\sqrt{\tfrac{6}{7}}-60\sqrt{\tfrac{5}{7}}=\frac{\sqrt{6}-\sqrt{5}}{\sqrt{7}}\times 60=4{\cdot}839951$ the 2d

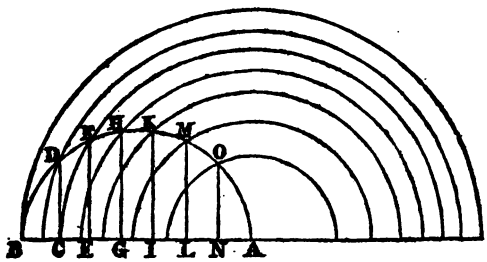
$60\sqrt{\tfrac{5}{7}}-60\sqrt{\tfrac{4}{7}}=\frac{\sqrt{5}-\sqrt{4}}{\sqrt{7}}\times 60=5{\cdot}353518$ the 3d

$60\sqrt{\tfrac{4}{7}}-60\sqrt{\tfrac{3}{7}}=\frac{\sqrt{4}-\sqrt{3}}{\sqrt{7}}\times 60=6{\cdot}076516$ the 4th

$60\sqrt{\tfrac{3}{7}}-60\sqrt{\tfrac{2}{7}}=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{7}}\times 60=7{\cdot}207871$ the 5th

$60\sqrt{\tfrac{2}{7}}-60\sqrt{\tfrac{1}{7}}=\frac{\sqrt{2}-\sqrt{1}}{\sqrt{7}}\times 60=9{\cdot}393480$ the 6th

$60\sqrt{\tfrac{1}{7}}-60\sqrt{\tfrac{0}{7}}=\frac{1}{\sqrt{7}}\times 60=22{\cdot}677870$ the 7th



" The Construction will be thus.—Divide the radius AB of the given circle into 7 equal parts; and at the points of division erect perpendiculars meeting the circle, described on AB as a diameter in D, F, H, K, M, O; then with the centre A, and radii AO, AM, AK, &c, circles being described, the thing is done.—For, by the nature of the circle, the squares of the chords or radii, AO, AM, AK, &c, are as the versed sines AN, AL, AI, &c.

" SCHOLIUM. It is evident that the above method of calculation and construction will both hold true also when the shares are unequal in any proportion; by using the respective proportional numbers in the former, and dividing the radius AB in the same proportion in the latter.

Very soon after the first publication of the solution of the Diary question as above, it was seized by a shameful plagiary, in the person of a Mr. Samuel Clark, who had commenced, after mine, a republication of the Diary questions, under the title of 'The Diarian Repository.' When this

editor arrived at the question above referred to, he copied my solution, and very inconsistently ascribed it to Mr. Hawney before mentioned, though it was manifest to every geometrician that no two constructions could be more unlike one another, as the one employed seven different circles, and as many separate constructions, in effecting what the other accomplished by only one single circle and construction. From the many gross errors, and the numerous omissions and absurdities in that 'Repository of Errors,' as it was commonly called, it soon fell under the necessity of coming to a sudden and premature end.

With respect to the other curious and kindred problem, that of dividing a given circle into any number of parts, that may be all mutually equal, both in area and perimeter, some account of its rise has been already given in the first volume of these tracts, at pa. 254. It was first anonymously proposed in the year 1774, as a curious paradoxical problem, but unaccompanied by the least hint or intimation of any mode of solution whatever. It was indeed announced by the proposer expressly as a seeming paradox, but accompanied with the declaration that it nevertheless was capable of a strict geometrical solution. The problem remained however some time unanswered, being given up by all persons as a matter quite hopeless, and by most deemed in fact as little to be expected as the quadrature of the circle itself, to which it was thought to be nearly allied, and indeed dependent on it; for no person could imagine any other possible way of a circle being divided, even in idea, into any *number* of such parts, that might be equal both in area and perimeter, than by radii drawn from the centre to the points of equal divisions in the circumference. This was, in effect, reducing the problem to this other, of dividing the circumference in *any* proposed number of equal parts, which was deemed on all hands a thing impossible to be effected. After some time no person thought any more of the matter, but as a thing never to be accomplished; and so I believe it might have remained to this day, but for the occurrence of some

such accident as that which actually led myself into the train of thought which soon ended in the complete solution. The construction I first inserted in the Critical Review, as before mentioned in the first volume of these Tracts; next it was introduced into my first or quarto volume of Tracts, published in the year 1786, accompanied with a short account of its rise, and a considerable improvement of it, by rendering the property general for the division into all ratios of parts, equal or unequal, and extending the same to all ellipses, as well as circles. After which I have usually been in the habit of introducing it into my Dictionary, and the more common elementary books on mensuration, &c.

Lastly, finding the two constructions introduced, by my friend Mr. Leslie, the ingenious and learned mathematical professor in the university of Edinburgh, into the first edition of his Geometry, published in 1809, both together in pages 222 and 223: as these problems were rather of an uncommon nature, I did think some mention might have been made of their origin, or the circumstances that have attended them; and I hinted as much to my ingenious friend. In consequence of which, probably, I find that the learned author has, in the 2nd edition of his work, separated those two constructions, placing one among the elements at p. 181, and the other among the notes at p. 432, accompanied with the note, that it was the result of a 'principle briefly *suggested* by Mr. Lawson, and afterwards explained and demonstrated in Dr. Hutton's Mathematical Tracts.' This change and announce seemed to make the matter rather worse than before, as it appeared less unfriendly, or less uncivil, to omit noticing a fact entirely, than to mis-state it. For, certain it is, that Mr. Lawson never *suggested* any principle or extension, nor any mode of solution whatever; the discovery having been made and published by myself alone.

FINIS.

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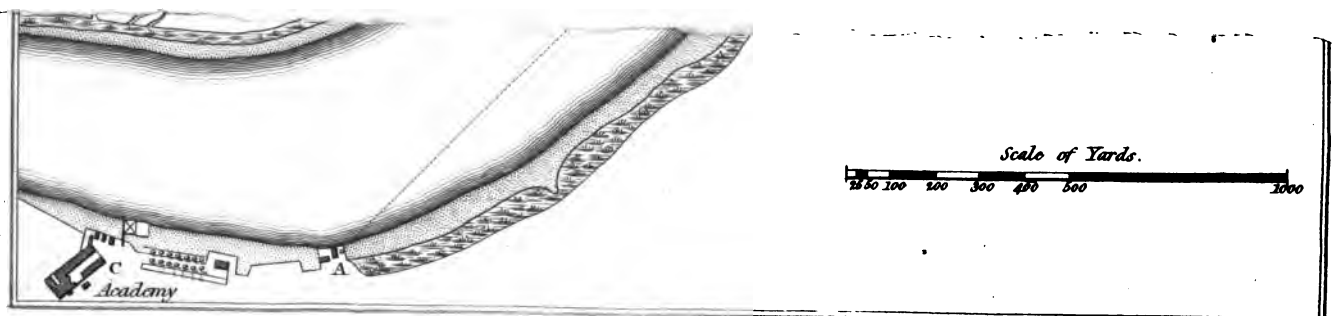
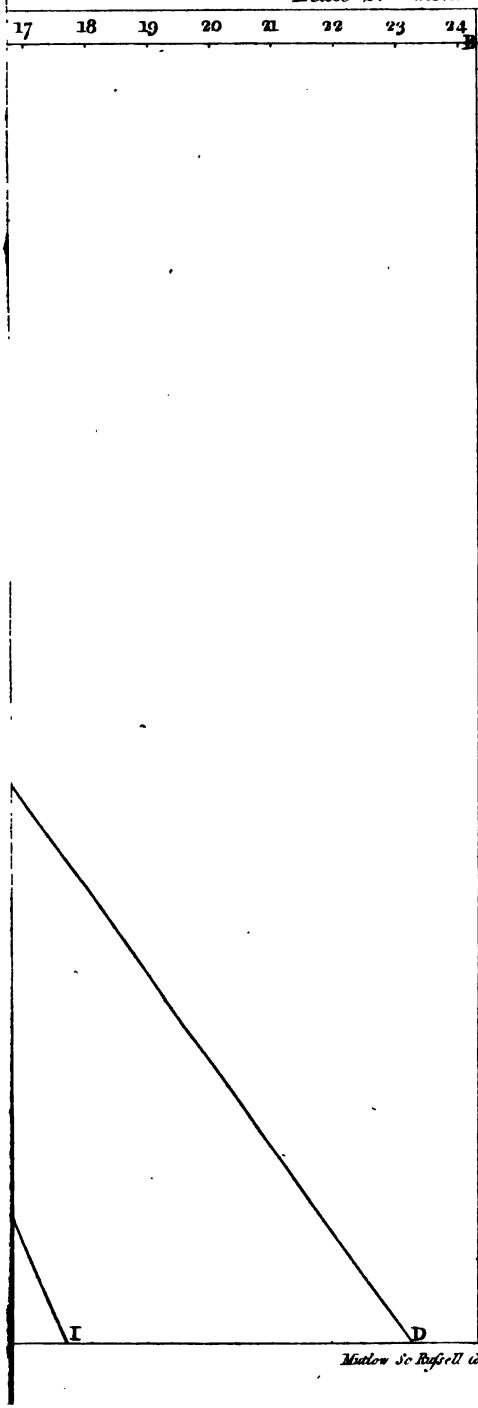


Mudrow Sc. Bagwell Ca.

Plate 2. the end.



Mutlow Sc Russell ex.

Plate III. the end

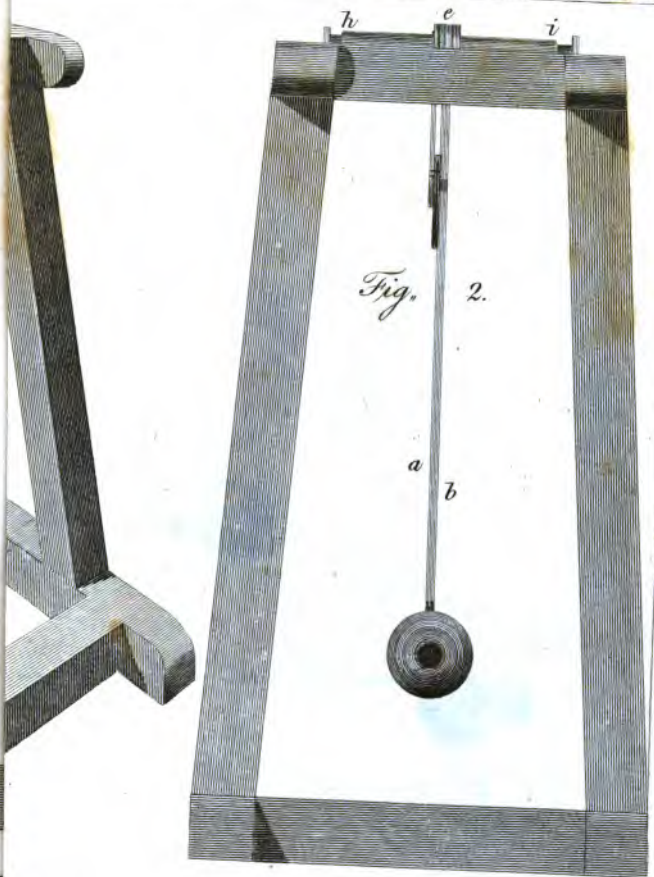
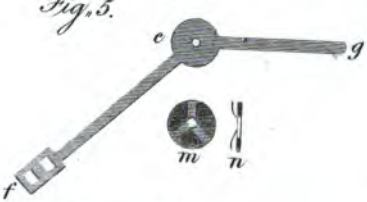


Fig. 5.



Scale of Feet and Inches.